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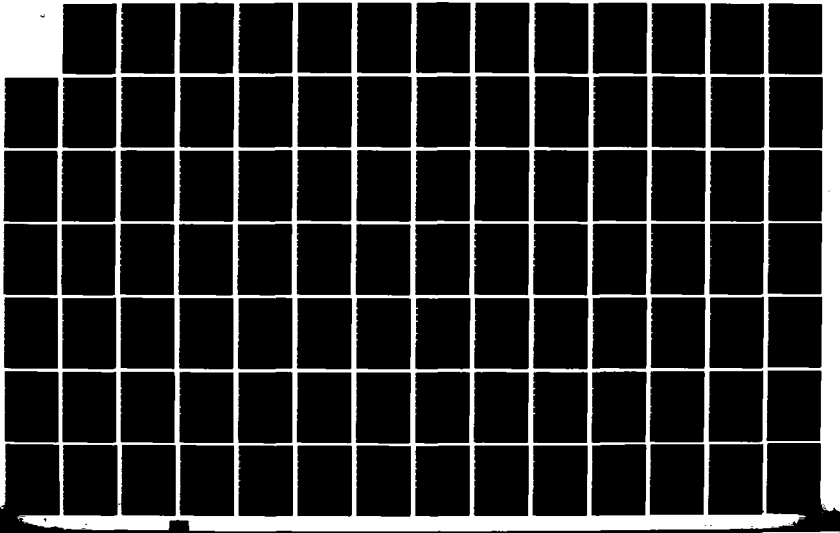

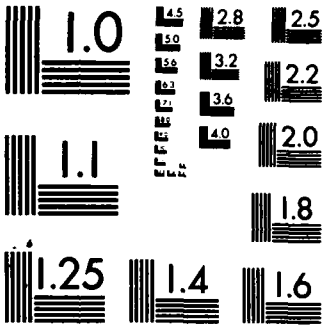

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AN APPLICATION OF THE H-FUNCTION TO CURVE-FITTING AND DENSITY ESTIMATION

THESIS

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AN APPLICATION OF THE H-FUNCTION TO CURVE-FITTING AND DENSITY ESTIMATION

THESIS

Presented to the Faculty of the School of Engineering
of the Air Force Institute of Technology
Air University
In Partial Fulfillment of the
Requirements for the Degree of
Master of Science in Operations Research

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Abstract

The H-function is the most general special function, encompassing as specific cases many mathematical functions and nearly every continuous statistical distribution defined over positive x. A general procedure is developed to estimate the parameters of the H-function which gives the best fit to a set of data. The technique creates a system of nonlinear equations from the method of moments and uses Powell's quasi-Newton hybrid algorithm to solve the equations. A computer program, which can accept both raw data or previously calculated moments, implements the general process. Several new theoretical results are also presented.

AN APPLICATION OF THE H-FUNCTION TO CURVE-FITTING AND DENSITY ESTIMATION

I Introduction

Analysts cannot effectively work with raw data. Before statistical analysis can be accomplished, data must be summarized in a convenient form. The most common way of achieving this is to fit data with the best function or statistical distribution. For example, if the analyst wants to model the time between arrivals to a queue for a computer simulation, he infers the true probability distribution from a random sample of observations. This process of statistical inference usually involves estimating both the form and the parameters of the distribution. Only after the best form has been found can the analyst begin to use standard statistical tools such as confidence intervals and hypothesis tests.

Background

Currently, analysts use the following procedure when attempting to fit mathematical functions or statistical distributions to data:

1. Plot the data,
2. Hypothesize a particular type of function,
3. Estimate the parameters, and
4. Test for goodness-of-fit using an appropriate test.

This procedure has two major shortcomings. First, the analyst must test each specific function separately. For example, one must perform two individual tests if the data is suspected of being from either a gamma distribution or a half-normal distribution. Second, several distributional forms and many combinations of parameters may fit the data. In the same example, if both goodness-of-fit tests fail to reject the hypothesized distribution, then neither can be eliminated as the true population distribution. Furthermore, there is no chance of finding the distribution which best fits the data unless all possible distributions are tested.

To solve these problems, analysts can use a general special function, called the H-function, because it includes many mathematical functions and statistical distributions as special cases. An analyst could simultaneously consider the special cases by simply fitting the H-function to the data. Although this idea seems logical, the H-function has never been applied to the current procedure for curve-fitting because of the newness and difficulty of the H-function theory.

Objective

The theory has progressed enough for an application of the H-function. Special theorems allow the application to curve-fitting without a full understanding of the complex H-function theory. Therefore, the overall objective of this

research is to develop an efficient and effective method to apply the H-function to the current procedure for curve-fitting and density estimation. To reach this objective, two main goals need to be accomplished.

First, a method of estimating the parameters of the H-function has to be found. The parameters uniquely define the H-function and therefore, with knowledge of the parameters, the H-function can be explicitly evaluated and graphed.

At this time, a subtlety needs to be discussed. The analyst may also know that the data comes from a statistical distribution. Because the method estimates parameters based on a finite number of data points, the parameters that appear in the H-function may not exactly define a statistical distribution. For example, a parameter required to be 1 for a chi-square distribution may be 1.01 when estimated. The other estimates may also be slightly off.

However, the estimates do not need to be reevaluated. The H-function found will fit the data better than any forced chi-square distribution. If necessary, the analyst should think of the unnamed H-function as a nearly chi-square distribution and proceed with further analysis given this new information.

The second main goal is to determine the efficiency and effectiveness of the H-function curve-fitting procedure. This part of the research is somewhat subjective as can be

seen from the following definitions of efficiency and effectiveness.

Efficiency is measured in terms of not only the number of separate tests required, but also the difficulty of each test. Because the H-function is a general function, the application of the H-function reduces the number of separate tests required. Since the H-function method can be implemented on the computer, curve-fitting will not be difficult. Therefore, the new technique should be efficient.

Effectiveness is measured in terms of the ability of the method to find the "best" function. Analysts have many techniques they use to compare functions in order to determine which one "best" fits the data. All involve some measure of the error between the proposed function and the data. Examples include the absolute distance, the maximum absolute distance, and the square of the distance. Because of its common use, the estimated mean squared error will be our criterion for measuring the closeness of the H-function to the data. A more formal definition follows in Chapter 6.

Three of the problems which influence effectiveness are the sensitivity to the number of data points, the inaccuracy of higher degree moments, and the necessity for an initial guess of the parameters. The size of the sample is a problem for all statistical methods. For the more common cases, only four to six moments have to be found. The third problem, involving the initial guess, can be controlled by

checking it against a set of H-function convergence conditions. Effectiveness should increase because the nonlinear solution method results in accurate convergence most of the time.

Scope

The H-function is only applicable for continuous functions defined over positive values of x. This is not as serious a restriction as it first appears. Methods exist which can fold a symmetric distribution and move its axis of symmetry. Therefore, distributions like a normal or Student's t can be evaluated with the H-function as half-normal or half-student. Such transformations are not the subject of this thesis. Distributions such as the half-normal will only be analyzed directly.

The H-function also can be designated with a certain order. Order is defined as the sum of the number of gamma terms in the definition of the H-function. This definition will be seen later in Chapter 2. For programming purposes, the highest order covered in this effort is five. Again, this is not a serious restriction. In fact, most known statistical distributions can be described by an H-function with order one or two. For more advanced mathematical functions such as arcsin or arctanh, the H-function still only needs to be of order four.

Overview

Chapter 2 contains a general discussion of the H-function including the definition, some special properties, and many special cases. A deliberate attempt is made to avoid discussion of as much of the complex theory as possible.

Chapter 3 presents the results of the curve-fitting literature review. It concludes with the selection of the appropriate parametric curve-fitting method to use with the H-function, the method of moments.

The method of moments produces a system of nonlinear equations that needs to be solved. The system is nonlinear because each equation involves products and quotients of gamma functions where the unknowns are in the arguments. In Chapter 4, a historical survey of nonlinear solution techniques is given. The technique known as M.J.D. Powell's hybrid algorithm is selected. This algorithm is already available on the AFIT CDC Cyber 750 computer in an IMSL routine called ZSPOW.

Chapter 5 discusses the development of a computer program that accomplishes the main objective of the research. The program estimates the parameters of the H-function from a set of univariate data or paired data after the data has been converted to moments. Once the parameters are estimated, the H-function can be explicitly evaluated and graphed. The H-function is evaluated by placing the

estimated parameters into a program which has the capability of determining the value of the H-function for any $x > 0$ [7:Appendix B]. The H-function will be evaluated at the same x values as the data points. This will be the longest chapter since it summarizes the complete methodology of the thesis effort.

Chapter 6 describes graphs which contain the estimated H-function, the actual data points, and the measure of the fit of the H-function to the data. Also, this chapter summarizes the measure of fit for each graph in tabular form.

Finally, we reach conclusions about the efficiency and effectiveness of the H-function curve-fitting procedure. Also in the final chapter, new findings are highlighted and further studies are recommended.

II The H-function

Definition

The H-function is the most general special function, encompassing as special cases most of the other special functions and elementary functions of mathematics and nearly every continuous statistical distribution defined over positive x.

The H-function is defined by either of the two forms:

$$H(z) = H^{m\,n}_{p\,q}\left[z:\{(a_i,A_i)\},\ i=1,\ldots,p;\ \{(b_j,B_j)\},\ j=1,\ldots,q\right]$$

$$= \frac{1}{2\pi i}\int_{C_1} \frac{\prod_{j=1}^{m}\Gamma(b_j+B_j s)\prod_{i=1}^{n}\Gamma(1-a_i-A_i s)}{\prod_{i=n+1}^{p}\Gamma(a_i+A_i s)\prod_{j=m+1}^{q}\Gamma(1-b_j-B_j s)}\, z^{-s}\, ds \qquad (2.1)$$

$$= \frac{1}{2\pi i}\int_{C_2} \frac{\prod_{j=1}^{m}\Gamma(b_j-B_j s)\prod_{i=1}^{n}\Gamma(1-a_i+A_i s)}{\prod_{i=n+1}^{p}\Gamma(a_i-A_i s)\prod_{j=m+1}^{q}\Gamma(1-b_j+B_j s)}\, z^{s}\, ds$$

where z and all a_i and b_j are real or complex numbers, all A_i and B_j are positive real numbers, and m, n, p, and q are integers such that $0\le m\le q$ and $0\le n\le p$. Empty products are defined to be unity (1). The path of integration, C_1, is a contour in the complex s-plane from $w-i\infty$ to $w+i\infty$, such that all Left Half-Plane (LHP) poles of $\prod_{j=1}^{m}\Gamma(b_j+B_j s)$ lie

to the left of C_1 and all Right Half-Plane (RHP) poles of $\prod_{i=1}^{n}\Gamma(1-a_i - A_i s)$ lie to the right. Similarly, C_2 is a contour running from $v-i\infty$ to $v+i\infty$, such that all RHP poles of $\prod_{j=1}^{m}\Gamma(b_j - B_j s)$ lie to the right of C_2 and all LHP poles of $\prod_{i=1}^{n}\Gamma(1-a_i + A_i s)$ lie to the left (26:2-3; 37:195; 7:32).

We will use the first equation in Eq (2.1) as our definition, although the two definitions are equivalent. When there is little chance of confusion, we will often abbreviate the H-function as either

$$H_{p\,q}^{m\,n}(z) \quad \text{or} \quad H_{p\,q}^{m\,n}[z:\{(a_i, A_i)\}; \{(b_j, B_j)\}].$$

We will define the order of an H-function as p+q. This represents the number of pairs of parameters $\{(a_i, A_i)\}$ and $\{(b_j, B_j)\}$ where each pair represents a gamma function in the integrand in Eq (2.1).

Convergence Conditions

For our purposes, the H-function defined in Eq (2.1) is valid if (26:3; 7:72):

$$D = \sum_{i=1}^{n} A_i + \sum_{j=1}^{m} B_j - \sum_{j=m+1}^{q} B_j - \sum_{i=n+1}^{p} A_i > 0$$

and a C_1 line can be placed between the LHP and RHP poles. More stringent conditions are developed later in Chapter 5 when D=0. When D<0, the H-function is not defined because the infinite sum which can represent it does not converge.

Properties

Although we will rarely use the following properties in this thesis, we will state them for the reader's future use (37:196; 26:4; 7:33-34):

Reciprocal Property.

$$H_{p\,q}^{m\,n}\left[\frac{1}{z}:\{(a_i, A_i)\};\ \{(b_j, B_j)\}\right]$$

$$= H_{q\,p}^{n\,m}\left[z:\{(1-b_j, B_j)\};\ \{(1-a_i, A_i)\}\right]$$

Argument to a Power Property.

$$H_{p\,q}^{m\,n}\left[z^c:\{(a_i, A_i)\};\ \{(b_j, B_j)\}\right]$$

$$= \frac{1}{c} H_{p\,q}^{m\,n}\left[z:\{(a_i, \frac{A_i}{c})\};\ \{(b_j, \frac{B_j}{c})\}\right], \text{ where } c>0$$

$$= -\frac{1}{c} H_{q\,p}^{n\,m}\left[z:\{(1-b_j, -\frac{B_j}{c})\};\ \{(1-a_i, -\frac{A_i}{c})\}\right],$$

where c<0

Multiplication by an Argument to a Power Property.

$$z^c\, H_{p\,q}^{m\,n}\left[z:\{(a_i, A_i)\};\ \{(b_j, B_j)\}\right]$$

$$= H^{m\,n}_{p\,q}\left[z:\{(a_i+A_ic,\ A_i)\};\ \{(b_j+B_jc,\ B_j)\}\right]$$

Reduction Property.

If a pair of "A" terms and a pair of "B" terms in an H-function are identical and one is in the numerator and the other is in the denominator, then it is equivalent to an H-function with a lower order. Specifically (26:4; 7:34-35):

$$H^{m\,n}_{p\,q}\left[z:\{(a_i,A_i)\};\ (b_1,B_1),\ \ldots,\ (b_{q-1},B_{q-1}),\ (a_1,A_1)\right]$$

$$= H^{m\ \ n-1}_{p-1\ q-1}\left[z:(a_2,A_2),\ \ldots,\ (a_p,A_p);\ \{(b_j,B_j)\}\right]$$

provided n>0 and q>m.

$$H^{m\,n}_{p\,q}\left[z:(a_1,A_1),\ \ldots,\ (a_{p-1},A_{p-1}),\ (b_1,B_1);\ \{(b_j,B_j)\}\right]$$

$$= H^{m-1\ \ n}_{p-1\ q-1}\left[z:\{(a_i,A_i)\};\ (b_2,B_2),\ \ldots,\ (b_q,B_q)\right]$$

provided m>0 and p>n.

We also found another way in which the H-function can reduce to one of lower order. If any A_i or B_j is close enough to zero, that gamma term in the integrand of Eq (2.1) is essentially a constant. Thus,

$$H_{p\,q}^{m\,n}[z] \approx \Gamma(b_1)\, H_{p\;\;q-1}^{m-1\;\;n}[z:\{(a_i,A_i)\};\ (b_2,B_2),\ \ldots,\ (b_q,B_q)]$$

for $B_1 \approx 0$ and $m \geq 1$.

$$H_{p\,q}^{m\,n}[z] \approx \frac{1}{\Gamma(1-b_q)}\, H_{p\;\;q-1}^{m\;\;n}[z:\{(a_i,A_i)\};\ (b_1,B_1),\ \ldots,\ (b_{q-1},B_{q-1})]$$

for $B_q \approx 0$ and $m < q$.

$$H_{p\,q}^{m\,n}[z] \approx \Gamma(1-a_1)\, H_{p-1\;\;q}^{m\;\;n-1}[z:(a_2,A_2),\ \ldots,\ (a_p,A_p);\ \{(b_j,B_j)\}]$$

for $A_1 \approx 0$ and $n \geq 1$.

$$H_{p\,q}^{m\,n}[z] \approx \frac{1}{\Gamma(a_p)}\, H_{p-1\;\;q}^{m\;\;n}[z:(a_1,A_1),\ \ldots,\ (a_{p-1},A_{p-1});\ \{(b_j,B_j)\}]$$

for $A_p \approx 0$ and $n < p$.

Therefore, there are two ways in which the H-function can reduce to a lower order. Gamma terms in the integrand of Eq (2.1) could cancel in the numerator and denominator or they could reduce to constants. These reduction properties could be useful in allowing a less restrictive assumption of the values of m, n, p, and q when fitting the H-function to data.

Special Cases

Integral Transforms.

The Laplace (L_r) and Fourier (F_t) transforms of an H-function are also H-functions (37:199-201; 7:35):

$$L_r\left\{ H_{p\ q}^{m\ n}(cz) \right\} = \frac{1}{r} H_{p+1\ q}^{m\ n+1}\left[\frac{c}{r}:(0,1),\{(a_i, A_i)\};\ \{(b_j, B_j)\}\right]$$

Using the property of multiplication by an argument to a power, this becomes:

$$= \frac{1}{c} H_{p+1\ q}^{m\ n+1}\left[\frac{c}{r}:(1,1),\{(a_i+A_i, A_i)\};\ \{(b_j+B_j, B_j)\}\right]$$

By the reciprocal property, this can be rewritten as:

$$= \frac{1}{c} H_{q\ p+1}^{n+1\ m}\left[\frac{r}{c}:\{(1-b_j-B_j, B_j)\};\ (0,1),\ \{(1-a_i-A_i, A_i)\}\right]$$

$$F_t\left\{ H_{p\ q}^{m\ n}(cz) \right\} = \frac{1}{c} H_{q\ p+1}^{n+1\ m}\left[\frac{-it}{c}:\{(1-b_j-B_j, B_j)\};\ (0,1),\ \{(1-a_i-A_i, A_i)\}\right]$$

Mathematical Functions.

The following elementary mathematical functions can be expressed as H-functions (26:10,151-152; 7:39-41,124):

$$e^{-z} = H^{1\ 0}_{0\ 1}[z:;(0,1)]$$

$$z^b e^{-z} = H^{1\ 0}_{0\ 1}[z:;(b,1)]$$

$$\frac{1}{B} z^{\frac{b}{B}} e^{-z^{\frac{1}{B}}} = H^{1\ 0}_{0\ 1}[z:;(b,B)]$$

$$z^b = H^{1\ 0}_{1\ 1}[z:(b+1,1);\ (b,1)]$$

$$z^b\ (1-z)^{+a} = \Gamma(a+1)\ H^{1\ 0}_{1\ 1}[z:(a+b+1,1);\ (b,1)]$$

$$z^b\ (1+z)^{-a} = \frac{1}{\Gamma(a)}\ H^{1\ 1}_{1\ 1}[z:(b-a+1,1);\ (b,1)]$$

$$\sin(z) = \frac{\sqrt{\pi}}{2}\ H^{1\ 0}_{0\ 2}[\frac{z}{2}:;(\frac{1}{2},\frac{1}{2}),\ (0,\frac{1}{2})]$$

$$\sinh(z) = -\frac{i\sqrt{\pi}}{2}\ H^{1\ 0}_{0\ 2}[\frac{iz}{2}:;(\frac{1}{2},\frac{1}{2}),\ (0,\frac{1}{2})]$$

$$\cos(z) = \frac{\sqrt{\pi}}{2}\ H^{1\ 0}_{0\ 2}[\frac{z}{2}:;(0,\frac{1}{2}),\ (\frac{1}{2},\frac{1}{2})]$$

$$\cosh(z) = \frac{\sqrt{\pi}}{2}\ H^{1\ 0}_{0\ 2}[\frac{iz}{2}:;(0,\frac{1}{2}),\ (\frac{1}{2},\frac{1}{2})]$$

$$\arcsin(z) = -\frac{i}{4\sqrt{\pi}}\ H^{1\ 2}_{2\ 2}[iz:(1,\frac{1}{2}),\ (1,\frac{1}{2});\ (\frac{1}{2},\frac{1}{2}),\ (0,\frac{1}{2})]$$

$$\arcsinh(z) = \frac{1}{4\sqrt{\pi}}\ H^{1\ 2}_{2\ 2}[z:(1,\frac{1}{2}),\ (1,\frac{1}{2});\ (\frac{1}{2},\frac{1}{2}),\ (0,\frac{1}{2})]$$

$$\arctan(z) = \frac{1}{4}\ H^{1\ 2}_{2\ 2}[z:(1,\frac{1}{2}),\ (\frac{1}{2},\frac{1}{2});\ (\frac{1}{2},\frac{1}{2}),\ (0,\frac{1}{2})]$$

$$\operatorname{arctanh}(z) = -\frac{i}{4}\, H^{1\,2}_{2\,2}\left[iz:(1,\tfrac{1}{2}),\ (\tfrac{1}{2},\tfrac{1}{2});\ (\tfrac{1}{2},\tfrac{1}{2}),\ (0,\tfrac{1}{2})\right]$$

$$\log(1\pm z) = H^{1\,2}_{2\,2}\left[\pm z:(1,1),\ (1,1);\ (1,1),\ (0,1)\right]$$

$$\log(z) = \begin{cases} -H^{2\,0}_{2\,2}\left[z:(1,1),\ (1,1);\ (0,1),\ (0,1)\right] & 0<z\leq 1 \\ H^{0\,2}_{2\,2}\left[z:(1,1),\ (1,1);\ (0,1),\ (0,1)\right] & z>1 \end{cases}$$

It should be noted that the formulas given above are not consistent with those in Mathai and Saxena for arcsin(z), arctanh(z), and $\log(1\pm z)$. There were apparently typographical errors in their book. We believe that the correct formulas are given above. We verified these as the correct formulas by summing the residues of the H-functions given above. In each case we obtained the correct infinite series. The outlines of these proofs are provided in Appendix A.

We were able to prove a generalization to the logarithmic function log(z). It is given below and the proof is outlined in Appendix A.

$$\log(z) = \begin{cases} -u^2\, H^{2\,0}_{2\,2}\left[z:(1,u),\ (1,u);\ (0,u),\ (0,u)\right] & 0<z\leq 1 \\ u^2\, H^{0\,2}_{2\,2}\left[z:(1,u),\ (1,u);\ (0,u),\ (0,u)\right] & z>1 \end{cases}$$

A similar generalization applies to the power function z^b. It is stated below and the proof is a special case of the proof for the generalization of the Power Function probability density function (p.d.f.) in Appendix A.

$$z^b = u H^{1\ 0}_{1\ 1}[z:(ub+1,u);\ (ub,u)]$$

The H-function also includes as special cases many advanced mathematical functions. Since these will not be dealt with in this thesis, they are provided for the reader's benefit in Appendix B.

Statistical Distributions.

Consider a continuous random variable X whose probability density function (p.d.f.) is given by

$$f_X(x) = \begin{cases} k\, H^{m\ n}_{p\ q}(cx) & cx \in S \\ 0 & \text{otherwise} \end{cases}$$

where k and c are constants such that $\int_{-\infty}^{\infty} f_X(x)\,dx = 1$ and S is a subset of the positive real line for which $H^{m\ n}_{p\ q}(cx)$ is convergent. Then the random variable X is said to be an H-function variate or a random variable with an H-function distribution (37:200; 7:84).

Many common statistical distributions are special cases of the H-function distribution. These include (37:164, 202-207; 7:85-87,93-94):

Gamma p.d.f.

$$f(x|\theta,\phi) = \frac{\phi^{\theta}}{\Gamma(\theta)} x^{\theta-1} e^{-\phi x}$$

$$= \frac{\phi}{\Gamma(\theta)} H^{1\ 0}_{0\ 1}[\phi x:;(\theta-1,1)] \qquad x > 0$$

$$\theta,\phi > 0$$

Exponential p.d.f. (Gamma p.d.f. with $\theta = 1$)

(Weibull p.d.f. with $\theta = 1$)

$$f(x|\phi) = \phi e^{-\phi x}$$

$$= \phi H^{1\ 0}_{0\ 1}[\phi x:;(0,1)] \qquad x > 0$$

$$\phi > 0$$

Chi-Square p.d.f. (Gamma p.d.f. with $\theta = \frac{\theta'}{2}$ and $\phi = \frac{1}{2}$)

$$f(x|\theta') = [2^{\frac{\theta'}{2}} \Gamma(\frac{\theta'}{2})]^{-1} x^{\frac{\theta'}{2}-1} e^{-\frac{x}{2}}$$

$$= [2\,\Gamma(\frac{\theta'}{2})]^{-1} H^{1\ 0}_{0\ 1}[\frac{x}{2}:;(\frac{\theta'}{2}-1,1)] \qquad x > 0$$

$$\theta' > 0$$

Weibull p.d.f.

$$f(x|\theta,\phi) = \theta\phi x^{\theta-1} e^{-\phi x^{\theta}}$$

$$= \phi^{\frac{1}{\theta}} H^{1\ 0}_{0\ 1}[\phi^{\frac{1}{\theta}} x:;(1-\tfrac{1}{\theta},\tfrac{1}{\theta})] \qquad x > 0$$

$$\theta,\phi > 0$$

Rayleigh p.d.f. (Weibull p.d.f. with $\theta = 2$)

$$f(x|\phi) = 2\phi x\, e^{-\phi x^2}$$

$$= \sqrt{\phi}\, H^{1\,0}_{0\,1}\left[\sqrt{\phi}x :; \left(\tfrac{1}{2},\tfrac{1}{2}\right)\right] \qquad x > 0$$

$$\phi > 0$$

Maxwell p.d.f.

$$f(x|\theta) = \frac{4}{\theta^3\sqrt{\pi}}\, x^2\, e^{-\frac{x^2}{\theta^2}}$$

$$= \frac{2}{\theta\sqrt{\pi}}\, H^{1\,0}_{0\,1}\left[\frac{x}{\theta} :; \left(1,\tfrac{1}{2}\right)\right] \qquad x > 0$$

$$\theta > 0$$

Half-Normal p.d.f.

$$f(x|\theta) = \frac{2}{\sqrt{2\pi\theta}}\, e^{-\frac{x^2}{2\theta^2}}$$

$$= \frac{1}{\sqrt{2\pi\theta}}\, H^{1\,0}_{0\,1}\left[\frac{1}{\sqrt{2}\theta}x :; \left(0,\tfrac{1}{2}\right)\right] \qquad x > 0$$

$$\theta > 0$$

Beta p.d.f. of the first kind

$$f(x|\theta,\phi) = \frac{\Gamma(\theta+\phi)}{\Gamma(\theta)\Gamma(\phi)}\, x^{\theta-1}\, (1-x)^{\phi-1}$$

$$= \frac{\Gamma(\theta+\phi)}{\Gamma(\theta)}\, H^{1\,0}_{1\,1}\left[x : (\theta+\phi-1,1);\ (\theta-1,1)\right]$$

$$0 < x < 1$$

$$\theta, \phi > 0$$

Power Function p.d.f. (Beta p.d.f. with $\emptyset = 1$)

$$f(x|\theta) = \theta x^{\theta-1}$$

$$= \theta H^{1\ 0}_{1\ 1}\left[x:(\theta,1);\ (\theta-1,1)\right] \qquad 0 < x < 1$$

$$\theta > 0$$

Uniform p.d.f. (Beta p.d.f. with $\theta = \emptyset = 1$)

(Power Function p.d.f. with $\theta = 1$)

$$f(x) = 1$$

$$= H^{1\ 0}_{1\ 1}\left[x:(1,1);\ (0,1)\right] \qquad 0 < x < 1$$

Half-Cauchy p.d.f.

$$f(x|\theta) = \frac{2\theta}{\pi(\theta^2+x^2)}$$

$$= \frac{1}{\theta\pi} H^{1\ 1}_{1\ 1}\left[\frac{x}{\theta}:(0,\tfrac{1}{2});\ (0,\tfrac{1}{2})\right] \qquad x > 0$$

$$\theta > 0$$

Half-Student p.d.f.

$$f(x|\theta) = \frac{2\,\Gamma(\frac{\theta+1}{2})}{\sqrt{\theta\pi}\ \Gamma(\frac{\theta}{2})\ (1+\frac{x^2}{\theta})^{\frac{\theta+1}{2}}}$$

$$= \frac{1}{\sqrt{\theta\pi}\,\Gamma(\frac{\theta}{2})} H^{1\ 1}_{1\ 1}\left[\frac{x}{\sqrt{\theta}}:(\tfrac{1-\theta}{2},\tfrac{1}{2});\ (0,\tfrac{1}{2})\right] \qquad x > 0$$

$$\theta > 0$$

F p.d.f.

$$f(x|\theta,\emptyset) = \frac{\Gamma(\frac{\theta+\emptyset}{2})\ \theta^{\frac{\theta}{2}}\ \emptyset^{\frac{\emptyset}{2}}\ x^{\frac{\theta}{2}-1}}{\Gamma(\frac{\theta}{2})\ \Gamma(\frac{\emptyset}{2})\ (\theta x+\emptyset)^{\frac{\theta+\emptyset}{2}}}$$

$$= \frac{\theta}{\emptyset\Gamma(\frac{\theta}{2})\Gamma(\frac{\emptyset}{2})}\ H^{1\ 1}_{1\ 1}\left[\frac{\theta x}{\emptyset}:(-\tfrac{\emptyset}{2},1);\ (\tfrac{\theta}{2}-1,1)\right] \qquad x > 0$$

$$\theta,\ \emptyset > 0$$

Beta p.d.f. of the second kind

$$f(x|\theta,\emptyset) = (\emptyset/\theta)^{\theta}\ \frac{\Gamma(\theta+\emptyset)\ \ x^{\theta-1}}{\Gamma(\theta)\Gamma(\emptyset)\ \ (1+\frac{\emptyset x}{\theta})^{\theta+\emptyset}}$$

$$= \frac{\emptyset}{\theta\Gamma(\theta)\Gamma(\emptyset)}\ H^{1\ 1}_{1\ 1}\left[\frac{\emptyset x}{\theta}:(-\emptyset,1);\ (\theta-1,1)\right] \qquad x > 0$$

$$\theta,\ \emptyset > 0$$

These results are summarized in Table I (8:300).

We were able to prove a generalization to the above formulas for the Power Function p.d.f. and the Uniform p.d.f. This generalization was not previously known and the proofs are outlined in Appendix A.

Power Function p.d.f.

$$f(x|\theta) = \theta x^{\theta-1}$$

$$= u\theta H^{1\ 0}_{1\ 1}\left[x:(u(\theta-1)+1,u);\ (u(\theta-1),u)\right] \qquad 0 < x < 1$$

$$\theta,u > 0$$

TABLE I. Statistical Distributions

Distribution	H-function Distribution Parameters					
	m n p q	k	c	(a,A)	(b,B)
Gamma	1 0 0 1	$\theta[\Gamma(\theta)]^{-1}$	θ	---	$(\theta-1,1)$
Exponential	1 0 0 1	θ	θ	---	$(0,1)$
Chi-Square	1 0 0 1	$[2\Gamma(\theta'/2)]^{-1}$	$1/2$	---	$(\frac{\theta'}{2}-1,1)$
Weibull	1 0 0 1	$\theta^{1/\theta}$	$\theta^{1/\theta}$	---	$(1-\frac{1}{\theta},\frac{1}{\theta})$
Rayleigh	1 0 0 1	$\sqrt{\theta}$	$\sqrt{\theta}$	---	$(\frac{1}{2},\frac{1}{2})$
Maxwell	1 0 0 1	$2/(\theta\sqrt{\pi})$	θ^{-1}	---	$(1,\frac{1}{2})$
Half-Normal	1 0 0 1	$[\sqrt{2\pi}\theta]^{-1}$	$[\sqrt{2}\theta]^{-1}$	---	$(0,\frac{1}{2})$
Beta (first kind)	1 0 1 1	$\Gamma(\theta+\theta)/\Gamma(\theta)$	1	$(\theta+\theta-1,1)$	$(\theta-1,1)$
Power Function	1 0 1 1	θ	1	$(\theta,1)$	$(\theta-1,1)$
Uniform	1 0 1 1	1	1	$(1,1)$	$(0,1)$
Half-Cauchy	1 1 1 1	$[\theta\pi]^{-1}$	θ^{-1}	$(0,\frac{1}{2})$	$(0,\frac{1}{2})$
Half-Student	1 1 1 1	$[\sqrt{\theta\pi}\Gamma(\theta/2)]^{-1}$	$[\sqrt{\theta}]^{-1}$	$(\frac{1-\theta}{2},\frac{1}{2})$	$(0,\frac{1}{2})$
F	1 1 1 1	$\theta/[\theta\Gamma(\theta/2)\Gamma(\theta/2)]$	θ/θ	$(-\frac{\theta}{2},1)$	$(\frac{\theta}{2}-1,1)$
Beta (second kind)	1 1 1 1	$\theta/[\theta\Gamma(\theta)\Gamma(\theta)]$	θ/θ	$(-\theta,1)$	$(\theta-1,1)$

Uniform p.d.f. (Power Function p.d.f. with $\theta = 1$)

$$f(x) = 1$$

$$= u H^{1\ 0}_{1\ 1}[x:(1,u);\ (0,u)] \qquad 0 < x < 1,\ u > 0$$

We were also able to represent the Pareto p.d.f. as an H-function. This is another new finding not previously known. Again, the proof is outlined in Appendix A.

Pareto p.d.f.

$$f(x|\theta) = \theta x^{-\theta-1}$$

$$= \theta H^{0\ 1}_{1\ 1}[x:(-\theta,1);\ (-\theta-1,1)] \qquad x > 1,\ \theta > 0$$

This p.d.f. can also be generalized as above. The proof of the result is given in Appendix A.

Pareto p.d.f.

$$f(x|\theta) = \theta x^{-\theta-1}$$

$$= u\theta H^{0\ 1}_{1\ 1}[x:(1-u(1+\theta),u);\ (-u(1+\theta),u)] \qquad x > 1,\ \theta,u > 0$$

The Bessell p.d.f. and General Hypergeometric p.d.f. can also be expressed as H-functions. These are listed in Appendix B since we won't use them in this thesis. However, we will use the Bessel p.d.f. to verify and validate our computer program.

If a random variable has a p.d.f. that can be expressed as an H-function distribution, then its cumulative distribution function (c.d.f.) can be easily written as one minus another H-function. Specifically, if

$$f(x) = k\, H^{m\ n}_{p\ q}\left[cx:\{(a_i, A_i)\};\ \{(b_j, B_j)\}\right],$$

then (37:243; 7:102-107):

$$F(x) = 1 - \frac{k}{c}\, H^{m+1\ n}_{p+1\ q+1}\left[cx:\{(a_i+A_i, A_i)\},\ (1,1);\ (0,1),\ \{(b_j+B_j, B_j)\}\right]$$

Another powerful characteristic of H-function variates is that products, quotients, and rational powers of independent H-function variates are also random variables with H-function distributions. The exact formulas are lengthy and will not be presented here. The reader is referred to Springer (37:207-219) or Cook (7:90-92). In his dissertation, Cook presented a computer program that will evaluate these combinations and graph the p.d.f. and c.d.f. of the resulting H-function distribution.

Mellin Transformation and Moments

The Mellin transform (M_s) of an H-function is given as (37:198-199; 7:35):

$$M_s\{H(cx)\} = \frac{\prod_{j=1}^{m}\Gamma(b_j + B_j s)\prod_{i=1}^{n}\Gamma(1-a_i - A_i s)}{\prod_{i=n+1}^{p}\Gamma(a_i + A_i s)\prod_{j=m+1}^{q}\Gamma(1-b_j - B_j s)}\, c^{-s}$$

For continuous random variables defined over positive x, the moments about the origin are (37:201-202; 7:29,108-109):

$$\mu_r = E(X^r) = \int_0^\infty x^r f(x)dx$$
$$= M_{r+1}\{f(x)\}$$

Therefore, the moments of the H-function distribution are:

$$\mu_r = M_{r+1}\{kH^{m\,n}_{p\,q}(cx)\} = kM_{r+1}\{H^{m\,n}_{p\,q}(cx)\} = \frac{k}{c^{r+1}}I(r+1)$$

$$r=0,1,\ldots$$

where

$$I(s) = \frac{\prod_{j=1}^{m}\Gamma(b_j + B_j s)\prod_{i=1}^{n}\Gamma(1-a_i - A_i s)}{\prod_{i=n+1}^{p}\Gamma(a_i + A_i s)\prod_{j=m+1}^{q}\Gamma(1-b_j - B_j s)}$$

The above formula is especially useful for finding the constant k in an H-function distribution. Using the fact that the zeroth moment of a statistical distribution must equal one,

$$\mu_o = 1 = M_1\{kH^{m\,n}_{p\,q}(cx)\} = \frac{k}{c}I(1)$$

or (7:109):

$$k = \frac{c}{I(1)} = c \; \frac{\prod_{i=n+1}^{p} \Gamma(a_i + A_i) \; \prod_{j=m+1}^{q} \Gamma(1-b_j - B_j)}{\prod_{j=1}^{m} \Gamma(b_j + B_j) \; \prod_{i=1}^{n} \Gamma(1-a_i - A_i)}$$

The moments of H-function distributions carry their usual statistical meaning. For example, the zeroth moment is the area under the p.d.f. over the appropriate range. The first moment is the mean of the distribution. The second moment is a function of the mean and variance of the distribution.

We could also define moments of a mathematical function f(x) defined over positive x as:

$$\mu_r = \int_0^\infty x^r \, f(x) \, dx \qquad (2.2)$$

When the function can be represented as an H-function, these moments could be found by the same formula as for statistical distributions. However, they would not always have the statistical meaning. The first moment of a function could not always be interpreted as the mean or balance point of that function. The definition in Eq (2.2) will be useful in Chapter 5 when we fit mathematical functions to data using the method of moments.

This concludes our discussion of the definition, properties, special cases, and moments of the H-function. We

next examine procedures to estimate the parameters of the H-function. Because the H-function includes both mathematical functions and statistical distributions as special cases, we needed to find a method of estimating the parameters of the H-function that would be applicable for both curve-fitting and density estimation. Nearly every method we found was formulated for density estimation, although some could also be applied to curve-fitting.

It appears that when trying to fit mathematical functions to data, analysts usually plot the data points and hope to find a pattern in the points that is recognizable as a special mathematical function. If the points do not exhibit a pattern, the analysts could still approximate the n data points with a polynomial of order 0 to order n-1.

Because of the lack of techniques for fitting mathematical functions to data and the abundance of techniques for density estimation, we concentrated our literature review on methods of density estimation. As noted previously, in some cases, these methods may also be used to fit mathematical functions to data.

III Methods of Density Estimation

Methods of probability density estimation can be generally classified as parametric or nonparametric. Parametric density estimation techniques usually assume that the form of the distribution from which the data were taken is known. Alternatively, they may estimate the form or class of the probability density function (p.d.f.) from the data. In contrast, nonparametric density estimation techniques are not concerned with the form of the distribution, before or after the data are taken.

Generalized Families

Although parametric approaches generally require an assumption of the form of the unknown p.d.f., that assumption is not always as restrictive as it appears. There are several generalized families which include many density functions as special cases. Assuming that the sample came from a generalized family is not nearly as restrictive as assuming it came from a particular distribution.

The emphasis in the generalized approach is to leave the functional form of the unknown density as unspecified as possible and allow the data to indicate which special case of the family gives the best fit (41:13). This approach permits model and parameter estimation to be considered simultaneously. This seems to be beneficial because of the close relationship between a model and its parameters (12:1).

One of the best-known generalized families was proposed by Karl Pearson. This family, as it turned out, included many of the more common continuous univariate probability densities as members (41:5).

Pearson's system of frequency curves is generated by solutions to the differential equation

$$y' = \frac{x+a}{b_0+b_1x+b_2x^2}\, y$$

where a and the b's are constants (9:248-249; 37:255). The system consists of 12 types of curves and a set of rules for determining which curve best fits the data based on the first four moments (37:255). A fairly detailed development of the system is given by Elderton (13:38-127).

Special cases of the Pearson system of curves include the normal distribution, the chi-square distribution, Student's t distribution, the beta distributions (first and second kinds), and the Pareto distribution (9:249; 41:8). The gamma distribution can also be obtained after shifting the origin and making a transformation (41:9).

Another generalized family is the generalized gamma family defined by Stacy (38). It consists of probability density functions of the form:

$$f(x|a,d,p) = \frac{\frac{p}{(a)^d}}{\Gamma(\frac{d}{p})}\; x^{d-1}\; e^{-(x/a)^p}$$

where a, d, and p are parameters. This family includes the gamma, Weibull, Maxwell, and standard normal distributions as special cases (38:1187). Of course, the exponential and chi-squared distributions are also included in the family since they are special cases of the gamma distribution. Similarly, the Rayleigh distribution, a special case of the Weibull distribution, is also in the generalized gamma family. This family is equivalent to the first order H-function distribution $kH_{0\ 1}^{1\ 0}(cx)$.

Recently, Ramberg et al (33) proposed a four-parameter probability distribution whose percentile function R(p) is based on the generalization of Tukey's lambda function:

$$R(p) = \lambda_1 + \frac{[p^{\lambda_3} - (1-p)^{\lambda_4}]}{\lambda_2} \qquad 0 \leq p \leq 1$$

The density functions related to this percentile function can take on a variety of shapes, depending on the values of the λ_i. This distribution can represent, or at least approximate, the gamma, Weibull, normal, log-normal, and Student's t distributions (33:203, 206). The proposed distribution yields a good approximation to the data using the first four moments.

Clearly, the H-function is also a generalized family since it includes as special cases nearly every named continuous probability density defined over positive x. The

only named continuous densities that have not been shown to be H-functions are the log-normal and logistic distributions. On the other hand, no one has been able to show that these distributions are not H-functions.

As seen in Chapter 2 and Appendix B, the H-function also includes many named functions of mathematics as special cases. Thus, the assumption that the data came from an H-function could hardly be considered restrictive.

Still, that assumption makes fitting an H-function to data a parametric procedure. In this case, the assumed form of the distribution is an H-function with particular values for m, n, p, and q. Estimating the parameters of that H-function would simultaneously consider all of the densities which are special cases of the H-function and determine the H-function which best fits the data.

Parametric Estimation Techniques

Since the form of the distribution is assumed to be known in parametric density estimation approaches, the problem reduces to finding point estimates for the parameters of the p.d.f. Before discussing the various approaches to parameter estimation, however, we need to define certain desirable properties of estimators.

An estimate, $\hat{\theta}$, of an unknown parameter, θ, is said to be unbiased if $E(\hat{\theta})=\theta$ for all θ. "This implies that the sampling distribution of $\hat{\theta}$ is centered at the parameter θ.

That is, an unbiased estimator $\hat{\theta}$ is equal to θ on the average" (3:388).

If the estimate $\hat{\theta}$ converges in probability to θ, it is said to be consistent. Formally, if $\lim_{n\to\infty} \Pr\{|\hat{\theta}-\theta|<\xi\}=1$ for any $\xi>0$, then $\hat{\theta}$ is a consistent estimate of θ (18:235-236).

Consistent estimators are not necessarily unbiased and unbiased estimators are not necessarily consistent. Thus, neither property implies the other. But a consistent estimator with a finite mean value must tend to be unbiased in large samples (23:5).

An efficient estimator of θ is an unbiased estimator with minimum variance among all unbiased estimators. A measure of the efficiency of the estimator $\hat{\theta}_a$ is

$$\text{efficiency} = \frac{V(\hat{\theta})}{V(\hat{\theta}_a)}$$

where $V(\hat{\theta})$ is the minimum variance of all unbiased estimators and $V(\hat{\theta}_a)$ is the variance of $\hat{\theta}_a$. "An efficient estimator is sometimes called a minimum variance unbiased estimator" (3:388). An asymptotically efficient estimator is an estimator that becomes efficient as the sample size increases to infinity.

With these properties defined, we can proceed to discuss the various approaches to parameter estimation.

Method of Moments.

The method of moments was proposed by Karl Pearson to approximate data with a curve. This method involves equating the moments of the data with the moments of the distribution, creating as many equations as there are parameters to be estimated. The estimates are then obtained by solving these equations for the parameters (9:497-498; 13:12-37; 29:274-276). "This method often leads to comparatively simple computations in practice" (9:497).

However, one limitation of the method of moments is the unstable nature of the higher moments calculated from the data. If there are many unknown parameters, then higher order moments will be required to solve for the parameters.

> Karl Pearson has shown that "we might easily on a random sample reach a 7th or 8th moment having half or double the value it actually has in the general population. Constants based on these high moments will be practically idle. They may enable us to describe closely an individual random sample but no safe argument can be drawn from this individual sample as to the general population at large, at any rate so far as the argument is based on the constants depending on these high moments" (13:44).

This limitation led to the development of equations to correct the raw or grouped moments (9:360-362).

Another concern with the method of moments is the question of whether a finite number of moments can uniquely determine the distribution. Although there is a one-to-one

correspondence between the moment generating function and the distribution, unless the moment generating function is known, the moments, in general, do not uniquely determine the distribution function. This concern is referred to as the problem of moments (29:81).

Although estimates obtained with the method of moments are sometimes biased, we can often remove the bias with a simple correction and thus obtain an unbiased estimate (9:498). "In general, these estimates are consistent" (3:389). However, the asymptotic efficiency of the estimates is often considerably less than 1, which implies that they are not the "best" possible estimates from the efficiency point of view (9:498).

Method of Maximum Likelihood.

The concept of maximum likelihood was first introduced by R. A. Fisher in 1912 and applied to parameter estimation in 1921 (9:498).

There are two crucial assumptions of the method of maximum likelihood. First, the correct form of the equation must be known or assumed. Second, the data must be a representative sample from the whole range of situations about which the analyst wishes to generalize (10:7). The second assumption is common to all methods of density estimation. The first assumption, however, is characteristic only of parametric estimation techniques, although some parametric techniques allow a less restrictive assumption of the form

of the equation than others. This point was emphasized earlier in this chapter.

The method of maximum likelihood consists of determining the values $\hat{\theta}_1, \hat{\theta}_2, \ldots, \hat{\theta}_k$ which maximize the likelihood function with respect to $\theta_1, \ldots, \theta_k$. The likelihood function is defined as:

$$L = \prod_{i=1}^{n} f(x_i | \theta_1, \ldots, \theta_k)$$

where $f(x_i | \theta_1, \ldots, \theta_k)$ is the p.d.f. of X_i. This is equivalent to maximizing log L with respect to $\theta_1, \ldots, \theta_k$ since both L and log L are maximized at the same value. This is useful since log L is sometimes easier to maximize than L.

To maximize the likelihood function, an analyst usually differentiates L (or log L) with respect to each of the unknown parameters $\theta_1, \ldots, \theta_k$. These derivatives are then set equal to zero and the resulting system of equations is solved for $\theta_1, \ldots, \theta_k$. The solutions to these equations are the maximum likelihood estimators (23:35-74; 27:183-186). The primary difficulty with this method is that the system of equations often cannot be solved directly and the constants have to be found by numerical approximation (13:252).

Still, under some general conditions, maximum likelihood estimators are consistent, asymptotically normal, and asymptotically efficient (3:389). Although the estimates

are not necessarily unbiased, many times they can be modified so that they become unbiased (28:186). For these reasons, the method of maximum likelihood is the most widely used density estimation technique (41:13).

Method of Least Squares.

Another popular technique for fitting curves to data is the method of least squares. This method involves finding the constants of the assumed equation which minimize the square of the differences between the actual data values and the values predicted by the equation (23:75-91; 29:482-502).

Linear least squares is a well-developed technique that can be applied to any form of equation that can be reformulated through transformations into another equation that is linear in its coefficients. Daniel and Wood (10:19-23) suggest appropriate transformations to transform quite a few nonlinear equations into a model that is linear in its coefficients.

Nonlinear least squares estimation is a relatively new area developed to accomodate models which cannot be made linear through transformations. Several methods are available which use numerical techniques such as Gauss-Newton or steepest-descent to converge on a solution (10:9-10).

For the simple linear model and when general assumptions are made, the Gauss-Markov theorem states that the least squares estimators are the best (i.e. minimum variance) linear unbiased estimators of the unknown coefficients

in the model. Further, when the random errors in the model are normally distributed, the least squares estimates are maximum likelihood estimates and are of minimum possible variance (10:7).

Other Methods.

In the minimum chi-square method, the observations are grouped into c intervals and the values $\theta_1,\ldots,\theta_k$ are found which minimize

$$\chi^2 = \sum_{i=1}^{c} \frac{[n_i - np_i(\theta_1,\ldots,\theta_k)]^2}{np_i(\theta_1,\ldots,\theta_k)}$$

where n_i is the actual number of observations in interval i and $np_i(\theta_1,\ldots,\theta_k)$ is the predicted number of observations in the interval, regarded as a function of $\theta_1,\ldots,\theta_k$ (3:389). The asymptotic properties of minimum chi-square estimators are similar to those of maximum likelihood estimators (23:93). But as with some previously mentioned techniques, the equations are usually too difficult to be solved analytically and a numerical technique must be used. Further, the observations must be grouped, even when dealing with a continuous distribution, and it seems rather wasteful to impose an otherwise unnecessary grouping for estimation purposes (23:93).

In Bayesian statistics, θ is not regarded as an unknown constant, but as a random variable. Thus, it has a

probability density function, although this p.d.f. is unknown. The objective in Bayesian estimation is to combine any prior information about the distribution of θ with the random sample before estimating θ. Bayes' method is well formulated for a single θ. It involves multivariate distributions of $\theta_1,\ldots,\theta_k$ when a vector of parameters is considered.

If $g(\theta)$ is the p.d.f. of the parameter which expresses the prior information about θ and

$$f(x_1,\ldots,x_n|\theta) = \prod_{i=1}^{n} f(x_i|\theta)$$

is the joint p.d.f. of $X_1,\ldots,X_n$, given θ, then the posterior p.d.f. of θ, given the random sample, is

$$h(\theta|x_1,\ldots,x_n) = \frac{g(\theta)\ f(x_1,\ldots,x_n|\theta)}{\int g(\theta)\ f(x_1,\ldots,x_n|\theta)\ d\theta}$$

where the integration is performed over the possible values of θ. The posterior p.d.f. of θ represents the current knowledge about θ, incorporating the prior p.d.f. of θ and the random sample. Any measure of centrality of the posterior p.d.f., such as the mean, median, or mode, can be used as a point estimate of θ (29:339-351).

Another relatively simple way to estimate the form of the distribution is to use the graphical method. With this approach, the points of the empirical (i.e. sample) cumulative distribution function (c.d.f.) are plotted on

probability paper of the assumed type of distribution. If the points lie roughly in a straight line, then the correct form of distribution was assumed.

With some types of probability paper, a scale is provided to estimate the parameters of the p.d.f. (22:295-308). Alternatively, parameter estimates can be obtained using other techniques such as maximum likelihood.

A lesser-known technique of density estimation is the minimum-distance method. Given a distance function d(F,G) which measures how "far apart" two cumulative distribution functions F and G are, the minimum-distance estimate of Θ is the value of Θ which minimizes $d(F(x|\Theta),F_n(x))$, where $F_n(x)$ is the empirical c.d.f. Although intuitively appealing, the minimum-distance estimate is almost always difficult to find (29:287-288).

The Gram-Charlier type A series is sometimes used to approximate the p.d.f. of a distribution whose range is doubly infinite (i.e. $f(x)\geq 0$ for $-\infty<x<\infty$). It is based on the normal distribution and its derivatives and uses a series expansion involving Hermite polynomials to approximate the unknown p.d.f. (37:257-262; 9:222-227; 17:46-60). The Gram-Charlier type B series is based on the Poisson distribution and involves Poisson-Charlier polynomials (17:72-81).

In a recent dissertation, Hill (17) suggested several ways to estimate a p.d.f. if a finite number of moments or

the moment generating function is known. If the moment generating function is known and the function is continuous over the positive real line, then the p.d.f. can be found by finding the inverse Laplace transform of the moment generating function (17:92-106). Alternatively, the moment generating function could be used to obtain moments of the distribution and one of the series expansions, using the Gram-Charlier type A or B series or Laguerre series, could be used to approximate the p.d.f. (17:46-81). Hill also suggested using the moments to fit a curve of the Pearson family of frequency distributions (17:82-92).

Technique Selection

The H-function, defined in Chapter 2 as

$$H_{p\ q}^{m\ n}\left[x:\{(a_i,A_i)\},\ i=1,\ldots,p;\ \{(b_j,B_j)\},\ j=1,\ldots,q\right]$$

has 2(p+q) parameters to be estimated. For statistical distributions, two additional parameters are included to allow for scaling and to ensure that the H-function distribution integrates to one over the appropriate range. Thus, for H-function distributions, there are 2(p+q+1) parameters to be estimated. We therefore needed a method of estimating parameters that could produce estimates for a vector of parameters.

Method of Maximum Likelihood.

As noted in a previous section, the method of maximum likelihood can be used with a vector of parameters and is

widely used for this purpose. The maximum likelihood estimates also possess many desirable properties of estimators. Therefore, we attempted to obtain maximum likelihood estimates for the parameters of the H-function.

Let $X_1, X_2, \ldots, X_r$ be a random sample from the H-function distribution

$$k\,H_{p\,q}^{m\,n}\left[cx:\{(a_i,A_i)\},\ i=1,\ldots,p;\ \{(b_j,B_j)\},\ j=1,\ldots,q\right],$$

hereafter abbreviated as $kH_{p\,q}^{m\,n}(cx)$. Our objective is to obtain point estimates for the $2(p+q+1)$ parameters k, c, a_i and A_i $(i=1,\ldots,p)$, and b_j and B_j $(j=1,\ldots,q)$ using the method of maximum likelihood.

The likelihood function is simply the product of the individual densities. For a random sample of size r,

$$L(k,c,a_i,A_i,b_j,B_j|x_1,\ldots,x_r) = \prod_{h=1}^{r} kH_{p\,q}^{m\,n}(cx_h)$$

$$= \prod_{h=1}^{r}\left\{\frac{k}{2\pi i}\int_C \frac{\prod_{j=1}^{m}\Gamma(b_j+B_js)\ \prod_{i=1}^{n}\Gamma(1-a_i-A_is)}{\prod_{i=n+1}^{p}\Gamma(a_i+A_is)\ \prod_{j=m+1}^{q}\Gamma(1-b_j-B_js)}\,(cx_h)^{-s}\,ds\right\}$$

This function must be differentiated with respect to each of the parameters k, c, a_i, A_i, b_j, B_j. Differentiation with respect to k or c, while not trivial, is easy when compared to differentiation with respect to the other parameters.

Mathai and Saxena (26:19) give results, due to Buschman, for the Mellin transform of the partial derivatives of an H-function with respect to its parameters. These results imply that the partial derivatives may be brought through the contour integral and evaluated using the chain rule. For example,

$$\frac{\partial H_{O\ 1}^{1\ O}(x)}{\partial b} = \frac{1}{2\pi i}\int_C \Gamma(b+Bs)\ \Psi(b+Bs)x^{-s}\ ds$$

and

$$\frac{\partial H_{O\ 1}^{1\ O}(x)}{\partial B} = \frac{1}{2\pi i}\int_C s\ \Gamma(b+Bs)\ \Psi(b+Bs)x^{-s}\ ds$$

Consider the special case where k=c=m=q=1 and n=p=0. Then the likelihood function is:

$$L(b,B|x_1,\ldots,x_r) = \prod_{h=1}^{r} H_{O\ 1}^{1\ O}(x_h)$$

$$= \prod_{h=1}^{r}\left\{\frac{1}{2\pi i}\int_C \Gamma(b+Bs)\ (x_h)^{-s}\ ds\right\}$$

The maximum likelihood estimates for b and B are the solutions to the two equations:

$$\sum_{h=1}^{r}\left\{\frac{1}{2\pi i}\int_C \Gamma(b+Bs)\Psi(b+Bs)(x_h)^{-s}ds\right.$$

$$\left.\left[\prod_{j\neq h}\frac{1}{2\pi i}\int_C \Gamma(b+Bs)(x_j)^{-s}\ ds\right]\right\} = 0 \qquad (3.1)$$

and

$$\sum_{h=1}^{r} \left\{ \frac{1}{2\pi i} \int_C s\Gamma(b+Bs)\Psi(b+Bs)(x_h)^{-s} ds \right.$$

$$\left. \left[\prod_{j \neq h} \frac{1}{2\pi i} \int_C \Gamma(b+Bs)(x_j)^{-s}\, ds \right] \right\} = 0 \qquad (3.2)$$

Contour integrals are usually evaluated by summing the residues at the poles of the integrand. For the first contour integral in Eq (3.1), zero and all the negative integers are poles of order two. Therefore, it could be replaced by an infinite sum of residues, say $\sum_{J=0}^{\infty} g(x_h,b,B,J)$. The other contour integrals in Eq (3.1) are simply H-functions which equal

$$\frac{1}{B}(x_j)^{\frac{b}{B}} e^{-x_j^{\frac{1}{B}}}$$

For the first contour integral in Eq (3.2), zero and all the negative integers are poles of order two. Therefore, it could also be replaced as an infinite sum of residues, say $\sum_{J=0}^{\infty} u(x_h,b,B,J)$. Thus, Eqs (3.1) and (3.2) could be rewritten as:

$$\sum_{h=1}^{r} \left\{ \sum_{J=0}^{\infty} g(x_h,b,B,J) \left[\prod_{j \neq h} \left(\frac{1}{B}(x_j)^{\frac{b}{B}} e^{-(x_j)^{\frac{1}{B}}} \right) \right] \right\} = 0$$

and

$$\sum_{h=1}^{r}\left\{\sum_{J=0}^{\infty} u(x_h,b,B,J)\left[\prod_{j\neq h}\left(\frac{1}{B}(x_j)^{\frac{b}{B}} e^{-(x_j)^{\frac{1}{B}}}\right)\right]\right\} = 0$$

The maximum likelihood estimates are the values of b and B which satisfy the above equations.

It should be noted that the convenient assumptions k=c=m=q=1 and n=p=0 caused several simplifications. First, there are only two equations to be solved. Second, the integrand in each contour integral was relatively simple. With more complicated integrands, the evaluation of contour integrals by the sum of residues becomes more difficult. Finally, in the more general case, the H-functions in Eqs (3.1) and (3.2) could not be expressed in the closed form:

$$\frac{1}{B}(x_j)^{\frac{b}{B}} e^{-(x_j)^{\frac{1}{B}}}$$

Instead, each would be an infinite sum of residues, say $\sum_{J=0}^{\infty} 1(x_j,b,B,J)$. This would make Eq (3.1) of the form:

$$\sum_{h=1}^{r}\left\{\sum_{J=0}^{\infty} g(x_h,b,B,J)\left[\prod_{j\neq h}\left(\sum_{J=0}^{\infty} 1(x_j,b,B,J)\right)\right]\right\} = 0$$

Since no general results are known for the product of contour integrals or the product of infinite series, more research is required before maximum likelihood estimates for

the parameters of the H-function can be developed. Thus, the method of maximum likelihood cannot yet be used to obtain estimates for the H-function parameters.

Method of Moments.

The method of moments seems to be the next most widely used technique for density estimation, in spite of the concerns mentioned earlier. As seen in Chapter 2, the analytic moments of the H-function are simply products and quotients of gamma functions, where the argument of each gamma function is a linear combination of a pair of the parameters. These moments could be set equal to the moments of the data, creating as many equations as there are parameters to be estimated. These equations can, in theory, be solved to obtain estimates for the parameters. The method of moments, as applied to H-functions, will be fully derived in Chapter 5.

The other parametric density estimation techniques generally do not produce estimates as good as those from the method of moments. Although the method of linear least squares can also produce good estimates, the H-function cannot be transformed into an equation that is linear in its parameters.

IV Nonlinear Solution Technique

The method of moments produces a system of nonlinear equations that each involve products and quotients of gamma functions. We need to find a solution of these simultaneous equations. Numerical analysis techniques already exist that can accomplish this task. A survey of the most common techniques was conducted to determine which might best be applied to our specific problem. Three of the main considerations were convergence conditions, amount of calculation, and rate of convergence.

Fixed-Point Iteration

In fixed-point iteration, we consider two nonlinear equations in the form

$$\begin{aligned} f(x,y) &= 0 \\ g(x,y) &= 0 \end{aligned} \tag{4.1}$$

We rewrite these equations by taking an x out of the first equation and a y out of the second. This gives the equivalent form

$$\begin{aligned} x &= F(x,y) \\ y &= G(x,y) \end{aligned} \tag{4.2}$$

so that any root $(\bar{x},\bar{y})$ will solve both sets of equations. It should be noted that there are many ways to rewrite equations from the form of Eq (4.1) to the equivalent form of Eq (4.2).

The fixed-point iteration begins with an initial guess (x_o, y_o) and generates successive approximations from the recursive relationship [6:84]

$$\begin{aligned} x_{i+1} &= F(x_i, y_i) \\ y_{i+1} &= G(x_i, y_i) \end{aligned} \tag{4.3}$$

It is possible to accelerate the iteration process by using the most current information on x_{i+1} in the second equation of Eq (4.3). This produces

$$\begin{aligned} x_{i+1} &= F(x_i, y_i) \\ y_{i+1} &= G(x_{i+1}, y_i) \end{aligned}$$

which will converge provided the original iterative process in Eq (4.3) converges [5:446].

For the original iterative process, convergence occurs under the following sufficient (but not necessary) conditions:

1. F and G along with their first partial derivatives are continuous in a neighborhood about the root $(\bar{x}, \bar{y})$,
2. For all points in the neighborhood,

$$\left|\frac{\partial F}{\partial x}\right| + \left|\frac{\partial G}{\partial x}\right| \leq M < 1$$

$$\left|\frac{\partial F}{\partial y}\right| + \left|\frac{\partial G}{\partial y}\right| \leq M < 1$$

for some M, and

3. The initial approximation (x_o, y_o) is taken from the same neighborhood [6:84; 39:130; 35:223].

Moreover, if M is very small for all steps in the iteration (x_i, y_i), then the iteration converges quickly relative to the case when the magnitude of M is near one [39:130].

We define the order of convergence as a measure of the speed or rate of convergence. Order of convergence is the lowest value of n such that the n^{th}-derivative of g(x) evaluated at the solution $\bar{x}$ does not equal zero. For this reason, fixed-point iteration of the type $x = g(x)$ generally has first-order or linear convergence [39:84-85].

Newton's Method

Next, we discuss another fixed-point iteration of the type $x = g(x)$ which has second-order or quadratic convergence [44:145-146]. In this case, g(x) is chosen so that its first derivative vanishes at the solution $\bar{x}$.

The formula for a Newton iteration can be derived through a Taylor series expansion of f(x). When f(x) is twice continuously differentiable, then

$$f(x_{i+1}) = f(x_i) + \frac{f'(x_i)(x_{i+1}-x_i)}{1!} + \frac{f''(\xi)(x_{i+1}-x_i)^2}{2!}$$

where ξ lies between x_i and x_{i+1}. Suppose that x_{i+1} is chosen so that $f(x_{i+1})$ is nearly equal to zero and also that $(x_{i+1}-x_i)^2$ is sufficiently small so that the last term can be neglected. Then the above Taylor series simplifies to

$$0 = f(x_i) + f'(x_i)(x_{i+1}-x_i)$$

which can be solved for x_{i+1} to give the recursive relationship for Newton's Method as [44:132]

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)} \tag{4.4}$$

For the general form $x = g(x)$, the iterative process in Eq (4.4) will have second-order convergence if the derivative of $g(x)$ is zero at the solution $\bar{x}$. By differentiating,

$$g'(x) = \frac{f(x)\ f''(x)}{[f'(x)]^2}$$

and at $x = \bar{x}$,

$$g'(\bar{x}) = \frac{f(\bar{x})\ f''(\bar{x})}{[f'(\bar{x})]^2}$$

Since $f(\bar{x})=0$, the numerator is equivalent to zero. If the root $\bar{x}$ is simple, then $f'(\bar{x})$ is nonzero, $g'(\bar{x})$ is zero, and the iteration converges with order two. However, if multiple roots exist at $\bar{x}$, then L'Hospital's Rule must be used to show the convergence degenerates to first-order [39:85-89; 44:132-133].

If the derivative of $f(x)$ cannot be solved explicitly, then it can be solved using the approximation formula

$$f'(x) \approx \frac{f(x + \xi) - f(x)}{\xi} \tag{4.5}$$

for very small ξ [6:278].

To find the solution to the system of n nonlinear equations

$$\begin{aligned} f_1(\underline{x}) &= 0 \\ f_2(\underline{x}) &= 0 \\ &\vdots \\ f_n(\underline{x}) &= 0 \end{aligned} \tag{4.6}$$

the recursive relationship in Eq (4.4) can be expanded to

$$\underline{x}_{i+1} = \underline{x}_i - J(\underline{x}_i)^{-1} F(\underline{x}_i) \tag{4.7}$$

where $J(\underline{x}_i)$ is called the Jacobian matrix defined by

$$J(\underline{x}) = \begin{bmatrix} \frac{\partial f_1(\underline{x})}{\partial x_1} & \frac{\partial f_1(\underline{x})}{\partial x_2} & \cdots & \frac{\partial f_1(\underline{x})}{\partial x_n} \\ \frac{\partial f_2(\underline{x})}{\partial x_1} & \frac{\partial f_2(\underline{x})}{\partial x_2} & \cdots & \frac{\partial f_2(\underline{x})}{\partial x_n} \\ \vdots & \vdots & & \vdots \\ \frac{\partial f_n(\underline{x})}{\partial x_1} & \frac{\partial f_n(\underline{x})}{\partial x_2} & \cdots & \frac{\partial f_n(\underline{x})}{\partial x_n} \end{bmatrix}$$

evaluated at the i^{th} iteration of $\underline{x}$ and the evaluation of the equations in Eq (4.6) at $\underline{x}_i$ is $F(\underline{x}_i)$ [5:450].

The approximation formula for derivatives in Eq (4.5) can also be expanded to become

$$\frac{\partial f_j(\underline{x}_i)}{\partial x_k} \approx \frac{f_j(\underline{x}_i + \underline{e}_k \xi) - f_j(\underline{x}_j)}{\xi} \tag{4.8}$$

where ξ is very small and $\tilde{e}_k$ is a n-vector whose only non-zero entry is a one in the k^{th} row [5:456; 30:26].

For the recursive relationship in Eq (4.7), convergence occurs under the following sufficient (but not necessary) conditions:

1. $f_1, f_2, \ldots, f_n$ along with all derivatives through second order are continuous in a neighborhood about the root vector $\bar{x}$,
2. The Jacobian $J(\tilde{x}_i)$ does not vanish in the same neighborhood, and
3. The initial approximation $\tilde{x}_o$ is chosen sufficiently close to the root vector $\bar{x}$ [6:86; 5:449].

When the generalized Newton's method converges, it has second-order convergence if the roots are simple. But it is difficult to insure that the determinant of the Jacobian is not zero. Furthermore, the necessity to invert the Jacobian at each iteration requires many computations and thus simpler methods are needed in most cases [39:133].

Modified Newton's Method

For two nonlinear equations in two unknowns, the modification of Newton's method consists of applying the single-variable Newton method two times, once for each variable. Each time this is done, the other variable is assumed to be fixed. Succeeding approximations are then generated from the recursive relationship [39:136]

$$\begin{aligned} x_{i+1} &= x_i - \frac{f(x_i, y_i)}{f_x(x_i, y_i)} \\ y_{i+1} &= y_i - \frac{g(x_i, y_i)}{g_y(x_i, y_i)} \end{aligned} \tag{4.9}$$

which can be accelerated by using the most current information on x_{i+1} in the second equation of Eq (4.9) [6:88].

Note that generally we could use either f or g to calculate the new x and use the other function to calculate y. One of these choices will usually converge while the other diverges, depending on the exact problem. For n nonlinear functions in n unknowns, there are n! ways of choosing the n functions to find n unknowns. Often only one of these choices will converge and this is the main disadvantage of this method [39:136-143]. But if the correct combination is found, the convergence rate will be remarkably rapid and faster than linear convergence [6:87-89]. However, we need to find another simple method with a better chance for convergence to the root.

Steepest Descent Method

Next, we discuss a gradient search technique. In steepest descent, the recursive relationship for a general system of nonlinear equations is

$$\underline{x}_{i+1} = \underline{x}_i - J(\underline{x}_i)^T F(\underline{x}_i)$$

which is similar to Eq (4.7) except that the transpose of $J(\underline{x}_i)$ is used in place of the inverse [32:62-63]. Another way to look at steepest descent is to consider the problem of minimizing

$$M(\underline{x}_k) = \sum_{i=1}^{n} [f_i(\underline{x}_k)]^2 \qquad (4.10)$$

The value of $\underline{x}_k$ which causes $M(\underline{x}_k)$ in Eq (4.10) to equal zero will also solve the original set of n nonlinear equations [24:244-245]. Although it only promises linear convergence, "the method of Steepest Descent has been found to be an effective way of getting reasonably close to the solution" [32:63]. For this reason, hybrid algorithms based initially on the method of Steepest Descent followed by Newton's method can be very reliable for systems of nonlinear equations [14:36-37].

Levenberg-Marquardt Method

Such hybrid algorithms are quite commonly used today. When dealing with a system of n nonlinear equations, the recursive relationship for the Levenberg-Marquardt method is given by

$$\underline{x}_{i+1} = \underline{x}_i - H(\underline{x}_i)\ F(\underline{x}_i)$$

where $H(\underline{x}_i) = [J(\underline{x}_i)^T J(\underline{x}_i) + \lambda_i I]^{-1} J(\underline{x}_i)^T$ and $F(\underline{x}_i)$ is the evaluation of the equations in Eq (4.6) at $\underline{x}_i$ [32:63]. Note that as λ_i increases, the step vector $H(\underline{x}_i)$ tends toward the pure Steepest Descent vector. On the other hand, if λ_i=0, then the method reduces to Newton's method. By reducing λ_i systematically, the hybrid iteration combines the better features of both methods [32:63-64].

However, the problem of inverting and solving the Jacobian matrix at each iteration is still a major weakness because the Levenberg-Marquardt method still involves an inversion. The number of computations is excessive even for a computer. The solution to these problems involves algorithms that are known as quasi-Newton [4:577-578]. The idea of a quasi-Newton algorithm is to eliminate the calculations involved with the inversion of the Jacobian matrix [14: 38-45; 32:577-578].

Broyden's Procedure

Broyden's procedure can be used in an iteration method to avoid the inversion of the Jacobian matrix at every single iteration. The approximation to the inverse of the Jacobian matrix $A(\bar{X})$ is updated at each iteration using the formula

$$A(\bar{X}_{i+1}) = A(\bar{X}_i) + \frac{[(S_{i+1} - A(\bar{X}_i)Y_{i+1})S_{i+1}^T A(\bar{X}_i)]}{S_{i+1}^T A(\bar{X}_i) Y_{i+1}}$$

where $Y_{i+1} = F(\bar{X}_{i+1}) - F(\bar{X}_i)$ and

$$S_{i+1} = \bar{X}_{i+1} - \bar{X}_i$$ [5:455-460; 4:581].

Of course, for the first iteration the actual Jacobian matrix $J(\bar{X}_o)$ must be found explicitly or approximated using Eq (4.8). Then $J(\bar{X}_o)$ must be inverted once before Broyden's procedure can begin. After that, the procedure

always produces an approximation to the inverse of the Jacobian matrix $A(\underline{X}_i)$ which can be used to replace $J(\underline{X}_i)^{-1}$, the actual inverse of the Jacobian matrix. The recursive relationship is

$$\underline{X}_{i+1} = \underline{X}_i - A(\underline{X}_i)\ F(\underline{X}_i)$$

which is the same as Eq (4.7).

This procedure significantly reduces the number of arithmetic calculations and still provides super linear convergence [16:5-6]. Therefore, iterative methods with second-order convergence will approach second-order convergence when Broyden's procedure is used. Quadratic convergence will be obtained as the approximation to the inverse of the Jacobian matrix $A(\underline{X}_i)$ becomes better [5:456].

Powell's Method

The best combination of methods and procedures studied so far would be a quasi-Newton hybrid algorithm. This was found in an IMSL routine named ZSPOW which contains M.J.D. Powell's hybrid method (HYBRD1) for nonlinear equations [32:87-114]. ZSPOW not only includes the beneficial features of the Levenberg-Marquardt method, but also implements the calculation-saving strategy of Broyden's procedure [16:6-7; 30:45].

In a comparison of available software which solves systems of nonlinear equations, HYBRD1 had outstanding performance. Also, initial estimates of the parameters had

little effect on the convergence [16:24, 41-44]. All of this comes while still providing super linear convergence. Therefore, the method used in ZSPOW will nearly obtain second-order convergence.

In summary, the IMSL routine named ZSPOW has several advantages. First, ZSPOW is a hybrid routine which permits a bad initial guess of the root. Second, ZSPOW is quasi-Newton with a convergence rate that is nearly second-order. Next, the user needs to supply only a subroutine that contains the system of nonlinear equations that has to be solved. Finally, ZSPOW outputs error messages if the iteration does not make good progress.

We applied this numerical analysis technique to our system of nonlinear equations. The program described in the next chapter uses ZSPOW to produce accurate estimates for up to ten unknown variables from the same number of equations.

V Methodology

The method of moments has been selected as the most appropriate curve-fitting technique to estimate the parameters of the H-function. The method involves equating the appropriate number of analytic H-function moments with the same number of data moments. This will create as many equations as there are parameters to be estimated. As seen in the following derivation, the number of equations can be reduced by two through algebraic manipulation.

Generation of Equations

As discussed earlier in Chapter 2, the r^{th} moment, μ_r, of the H-function, $k H_{p\ q}^{m\ n}(cx)$, is defined by the following equation:

$$\mu_r = \frac{k}{c^{r+1}} I(r+1)$$

where

$$I(r+1) = \frac{\prod_{j=1}^{m} \Gamma(b_j+B_j+B_j r) \prod_{i=1}^{n} \Gamma(1-a_i-A_i-A_i r)}{\prod_{i=n+1}^{p} \Gamma(a_i+A_i+A_i r) \prod_{j=m+1}^{q} \Gamma(1-b_j-B_j-B_j r)}$$

Since I(r+1) has two parameters in the argument of every gamma function, each H-function moment equation will involve 2(p+q)+2 unknown parameters. The same number of equations

needs to be generated by setting μ_r equal to the data's r^{th} moment.

Since consecutive data moments should be used, the variable r could take on the values from 1 to 2(p+q)+2. Let the r^{th} moment of the data be represented by M_r. By solving each equation for k,

$$k = \frac{M_r \, c^{r+1}}{I(r+1)}$$

for r=1,2,...,2(p+q)+2. Since all the equations equal k,

$$k = \frac{M_1 \, c^2}{I(2)} = \frac{M_2 \, c^3}{I(3)} = \ldots = \frac{M_{2(p+q)+2} \, c^{2(p+q)+3}}{I(2(p+q)+3)} \qquad (5.1)$$

The adjacent equations in Eq (5.1) can be solved for c to give

$$c = \frac{M_r \quad I(r+2)}{M_{r+1} \quad I(r+1)}$$

for r=1,2,...,2(p+q)+1. Since all the equations equal c,

$$c = \frac{M_1 \; I(3)}{M_2 \; I(2)} = \frac{M_2 \; I(4)}{M_3 \; I(3)} = \ldots = \frac{M_{2(p+q)+1} \; I(2(p+q)+3)}{M_{2(p+q)+2} \; I(2(p+q)+2)} \qquad (5.2)$$

The adjacent equations in Eq (5.2) can be solved to give the following homogeneous equations:

$$\frac{M_i M_{i+2} \, [I(i+2)]^2}{(M_{i+1})^2 \; I(i+1)I(i+3)} - 1 = 0 \qquad (5.3)$$

for i=1,2,...,2(p+q).

Note that although this algebraic manipulation has reduced the number of equations from 2(p+q)+2 to 2(p+q), the analyst will still be required to calculate all 2(p+q)+2 data moments. Note also that if the zeroth moment is used, then Eq (5.3) will be evaluated for i=0,1,...,2(p+q)-1.

An example of this equation generation technique may be helpful at this time. For the generalized gamma distribution, the H-function is given as

$$kH_{0\ 1}^{1\ 0}\left[cx:(b,B)\right]$$

Since p+q=1, the method should generate four equations and then reduce the number of equations by two. Since $\Gamma(r+1)=\Gamma(b+B(r+1))$, the four equations produced by the method of moments will be

$$M_1 = \frac{k}{c^2}\Gamma(b+2B)$$

$$M_2 = \frac{k}{c^3}\Gamma(b+3B)$$

$$M_3 = \frac{k}{c^4}\Gamma(b+4B)$$

$$M_4 = \frac{k}{c^5}\Gamma(b+5B)$$

By solving for k and setting the equations equal to each other,

$$k = \frac{M_1\ c^2}{\Gamma(b+2B)} = \frac{M_2\ c^3}{\Gamma(b+3B)} = \frac{M_3\ c^4}{\Gamma(b+4B)} = \frac{M_4\ c^5}{\Gamma(b+5B)} \qquad (5.4)$$

The adjacent equations are then solved for c and set equal to each other to produce

$$c = \frac{M_1 \Gamma(b+3B)}{M_2 \Gamma(b+2B)} = \frac{M_2 \Gamma(b+4B)}{M_3 \Gamma(b+3B)} = \frac{M_3 \Gamma(b+5B)}{M_4 \Gamma(b+4B)} \tag{5.5}$$

The adjacent equations are again solved and the result is the two following homogeneous equations:

$$\frac{M_1 M_3 [\Gamma(b+3B)]^2}{(M_2)^2 \Gamma(b+2B)\Gamma(b+4B)} - 1 = 0$$

$$\frac{M_2 M_4 [\Gamma(b+4B)]^2}{(M_3)^2 \Gamma(b+3B)\Gamma(b+5B)} - 1 = 0$$

Once the two equations have been solved for the parameters b and B, then the estimates may be used in Eq (5.5) to solve for the value of c. After this has been done, k can be found using Eq (5.4).

As expected in the example, the method generated two equations in two variables. The two equations still require four data moments to be evaluated. To emphasize a point previously made, if M_0 through M_3 had been used instead, then the method would have resulted in the different homogeneous equations:

$$\frac{M_0 M_2 [\Gamma(b+2B)]^2}{(M_1)^2 \Gamma(b+B)\Gamma(b+3B)} - 1 = 0$$

$$\frac{M_1 M_3 [\Gamma(b+3B)]^2}{(M_2)^2 \Gamma(b+2B)\Gamma(b+4B)} - 1 = 0 \tag{5.6}$$

These two equations are in the form of Eq (5.3) when i=0 and when i=1 respectively.

Program Development

The general form of the equations in Eq (5.3) verifies the fact that a system of nonlinear equations needs to be solved. Even the simple case demonstrated in the example produced two simultaneous nonlinear equations. Chapter 4 concluded that the IMSL routine named ZSPOW would be the best numerical analysis technique to use for nonlinear systems of equations. Therefore, our program implements ZSPOW in order to find estimates for the desired parameters.

Phase One.

The actual program development was modularized. That is, the program was divided into four successive phases so that each was easier to solve. The first phase involved programs that could take perfect moments from known distributions and use ZSPOW to obtain estimates for the parameters. Only the a_i, A_i, b_j, and B_j need to be guessed initially. The values of c and k can be found using the formulas in Eq (5.2) and Eq (5.1) respectively.

The early programs in this phase could only handle one special case of the H-function at a time. This was because ZSPOW required a subroutine to contain the equations which needed to be solved and the early programs only used the actual equations like those in Eq (5.6). In addition, this phase's early programs could only output the results for the

one special H-function. However, these early runs did demonstrate the merit of the ZSPOW numerical analysis technique for finding accurate estimates of the parameters.

The later runs in phase one became more general. These programs could generate equations in the form of Eq (5.3) and output the results for the input values of m, n, p, and q. The only restriction was that p+q had to be smaller than six due to matrix dimension limitations. Our research was not expected to go past a 2^{nd} order H-function. Also, the input variable FLAG controlled whether the zeroth moment was used. Therefore, the user could generate the correct equations by using the proper input of FLAG.

This concluded the first phase programs. Accurate results were achieved with perfect moments for all 1^{st} and 2^{nd} order H-functions. At times when the initial guess of the parameters was far away from the proper number, ZSPOW would not converge on the expected root. This problem was corrected during the second phase.

Phase Two.

The second phase involved attempts to control the initial guess of the parameters. The first control consisted of checking the initial guess against a set of H-function convergence conditions (26:3; 7:72). Some of these conditions were mentioned previously in Chapter 2, but now we fully discuss the subject.

If a_i, b_j, A_i, B_j, m, n, p, and q hold their usual meaning in the definition of the H-function, then the new terms D, E, and L can be defined by the following equations:

$$D = \sum_{i=1}^{n} A_i + \sum_{j=1}^{m} B_j - \sum_{i=n+1}^{p} A_i - \sum_{j=m+1}^{q} B_j$$

$$E = \sum_{i=1}^{p} A_i - \sum_{j=1}^{q} B_j \qquad (5.7)$$

$$L = \sum_{j=1}^{q} b_j - \frac{q}{2} - \sum_{i=1}^{p} a_i + \frac{p}{2}$$

Chapter 2 also pointed out that all Left Half-Plane (LHP) poles of $\prod_{j=1}^{m} \Gamma(b_j+B_j s)$ must lie to the left of C_1 and all Right Half-Plane (RHP) poles of $\prod_{i=1}^{n} \Gamma(1-a_i-A_i s)$ must lie to the right of C_1. From this point on, we refer to the path of integration (C_1) as the ω line. Since there may be a significant distance between the right-most LHP pole and the left-most RHP pole, the ω line may be placed anywhere within the range defined by those two values. This distance will be referred to as the ω range where ω_{low} is the right-most LHP pole and ω_{high} is the left-most RHP pole.

With these definitions, a completely specified H-function represented by an infinite sum of residues does not converge under any of the following conditions:

Case 1. $D = 0$ $E < 0$ $L > (E)\,\omega_{low}$

Case 2. $D = 0$ $E > 0$ $L > (E)\,\omega_{high}$

Case 3. $D = 0$ $E = 0$ $L > 0$

Case 4. $D = 0$ $E = 0$ $L = 0$

Case 5. $D < 0$

Case 6. $\omega_{low} \geq \omega_{high}$

For all other cases, the H-function will converge.

This convergence check was placed in a program subroutine. After the program user makes an initial guess of the parameters, the completely specified H-function is checked against the convergence conditions. This forces the initial guess to meet convergence conditions and helps ZSPOW find the correct root. As a last check, the final estimate of the parameters determined by ZSPOW is also checked for convergence.

Up to this point, the user had been given the responsibility for the initial guess of the parameters. The second control on the initial guess involved supplying initial guess default values. Two requirements were built in. First, the default values were designed to always meet the initial convergence condition check. Second, the default guesses only take on values between zero and one. The reason for this may not be clear. For common distributions, when an H-function parameter is specified by an exact number, that number will take on the value 0, 1/2, or 1 (Table I). When a H-function parameter is represented by a variable, no

default guess will be appropriate for all possible values of the variable. Therefore, the default guess of the parameters is as good as possible.

Many runs on 1^{st} and 2^{nd} order H-functions were performed at this time to validate the use of the variables FLAG and GUESS. When FLAG = 0, the zeroth moment was used. When FLAG = 1, the moments began with the first moment. The same results were achieved on each run no matter how FLAG was set. Since we were concerned with the inaccuracy of higher degree moments, most future runs were performed with FLAG = 0. This is the suggested configuration for running the program. However, when the zeroth moment is inaccurate, the program user may wish to set FLAG = 1.

If GUESS = 0, the user supplied the initial guess of the H-function parameters. When GUESS = 1, the default initial guess was used. The default guess also performed as expected. If distributions had actual parameter values around the range (0,1), then GUESS = 1 converged to the correct root. If distributions were run that had actual parameter values too far from the (0,1) range, GUESS = 1 produced an error message. For example, Beta (Θ=2,$\emptyset$=10) will not run using the default guess because the initial estimate of small a=0.7 is not close enough to the actual value of a=11.0. Therefore, we suggest that the program user set GUESS = 0 and make an initial guess unless he thinks the unknown distribution has actual values for the

parameters near the range (0,1). Table II lists the distributions which may always be run with the variable GUESS set to one.

Table II. Distributions With Constant H-function Parameters

	(a,A)	(b,B)
Exponential	-	(0,1)
Rayleigh	-	(1/2,1/2)
Maxwell	-	(1,1/2)
Half-Normal	-	(0,1/2)
Uniform	(1,1)	(0,1)
Bessel	-	(0,1/2) (0,1/2)

All other runs with GUESS = 0 converged when a reasonable guess of the parameters was made.

In conclusion, the accurate convergence to the root by ZSPOW was enhanced by the convergence checker subroutine and the option for an initial default guess of the parameters. At this time, we had produced a program which could consistently fit more special functions than any previous general special function procedure.

Phase Three.

In all previous computer runs, exact moments had been used. In this respect, the computer program had been verified. By verification, we mean the program performed as it was expected to perform. Phase 4 would continue the verification process by insuring that error messages performed as desired. On the other hand, Phase 3 concentrated on validation. If the program was to be valid, it should not require perfect moments.

One way to input imperfect moments would be to add some error to each moment. A more reasonable approach might be to calculate the moments from raw data. But an analyst cannot always control the type of data he will receive. Therefore, Phase 3 added the capability to input the following four types of data:

Type 0. Previously calculated moments

Type 1. Univariate deviates

Type 2. Ordered pairs from a relative frequency

Type 3. Ordered pairs from a continuous function

The last three types were new and each needed a subroutine that could calculate moments from raw data.

For type one, the data's r^{th} moment was calculated by using

$$M_r = \sum_{i=1}^{n} \frac{(x_i)^r}{n}$$

where there are n univariate data points. For type two, the data's r^{th} moment was calculated by using

$$M_r = \sum_{j=1}^{m} (x_j)^r \Pr(X=x_j) \tag{5.8}$$

where there are m ordered pairs and $\sum_{j=1}^{m} \Pr(X=x_j) = 1$. For type three, the data's r^{th} moment is exactly represented by

$$M_r = \int_{-\infty}^{\infty} (x)^r f(x)\, dx$$

This formula may be approximated by using

$$M_r = \sum_{k=1}^{m} (x_k)^r f(x_k)\, \Delta x \tag{5.9}$$

where there are m ordered pairs, $\sum_{k=1}^{m} f(x_k)$ need not equal one, and Δx is the interval between the x_k values. Since Eqs (5.8) and (5.9) were similar and both involved ordered pairs, they were combined into a single subroutine.

Recall that the H-function is only applicable for continuous functions. This means that even though discrete distributions like the Poisson can generate data in the form of type one or two, the H-function will not be useful in these instances. Only univariate data from continuous distributions defined over positive x should be used in type

one. Similarly, only relative frequency data from continuous distributions defined over positive x should be used in type two.

At this stage, we generated data to validate the program. Type one data consisted of deviates generated from standard distributions. For example, the exponential deviates with Ø=2 were generated by IMSL using the formula

$$x = -[\ln(1-z)]/2$$

where z is a random number from a uniform (0,1) distribution.

Type two data was created by classifying the type one deviates into intervals. This produced ordered pairs (x,f(x)) where x is the midpoint of the interval and f(x) = Pr(X=x). The Pr(X=x) is the proportion of deviates within the interval. In the exponential example, the ordered pairs may look something like

(.5,.5)
(1.5,.25)
(2.5,.10)
(3.5,.06)
(4.5,.04)
(5.5,.03)
(6.5,.02)

The first ordered pair represents the fact that 50% of the exponential deviates were located within the interval (0,1). In truth, an analyst would never combine type one data into intervals to create type two data. By doing this,

he would only lose information. We go from type one to type two data only for the ease of generating ordered pairs of a relative frequency from a continuous distribution.

Finally, category three data was created by assuming the function was known. This produced ordered pairs (x,f(x)) which basically plot the function at a limited number of points. To keep the mathematical calculations simple, the function is evaluated so that Δx is the same between all x values. In the exponential example,

$$f(x) = 2e^{-2x}$$

and the ordered pairs would be created by observing the value of f(x) as x is incremented from 0.01 to 10 by 0.01.

We were careful that each type of data was generated from the same distributions and functions that had been used for verification. The moments calculated from the data by a moment generating program were compared to the perfect moments. The data moments were fairly close to the true moments but they became increasingly inaccurate for higher degree moments. Next, we validated the convergence of the phase three program with the inaccurate moments. The program converged to nearly the same root that had been observed during verification. Finally, the actual raw data was input into the program. The program always converged to the same root as had been seen with the inaccurate moments. This validated our program's ability to handle raw data.

In conclusion, the program now had the following features. The program could input either previously calculated moments or three types of raw data. If raw data was input, the program would generate the appropriate number of data moments. The program would then supply a default initial guess of the H-function parameters if the user did not supply his own. The completely specified H-function would next be checked against the convergence conditions. If they were not satisfied, the program would be terminated. If the convergence conditions were met, the program would continue by using ZSPOW to get final estimates of the H-function parameters. Once ZSPOW had finished, the final estimates were again tested against the convergence conditions. The final results were then output along with an error message if the last convergence test had failed. The program's flowchart can be seen in Figure 1.

The reason for the output even if the final estimate did not meet convergence conditions was to check if the inaccuracy of the moments had caused the problem. An example of this might be an estimated B=1.0 in the numerator and an estimated A=1.01 in the denominator. This would produce a D=-.01 in Eq (5.7) and therefore the H-function would not meet the convergence conditions. However, both estimates should probably equal one. The estimate of A may be off due to imperfect moments. The only way to find this type of error is to always output any final results of ZSPOW.

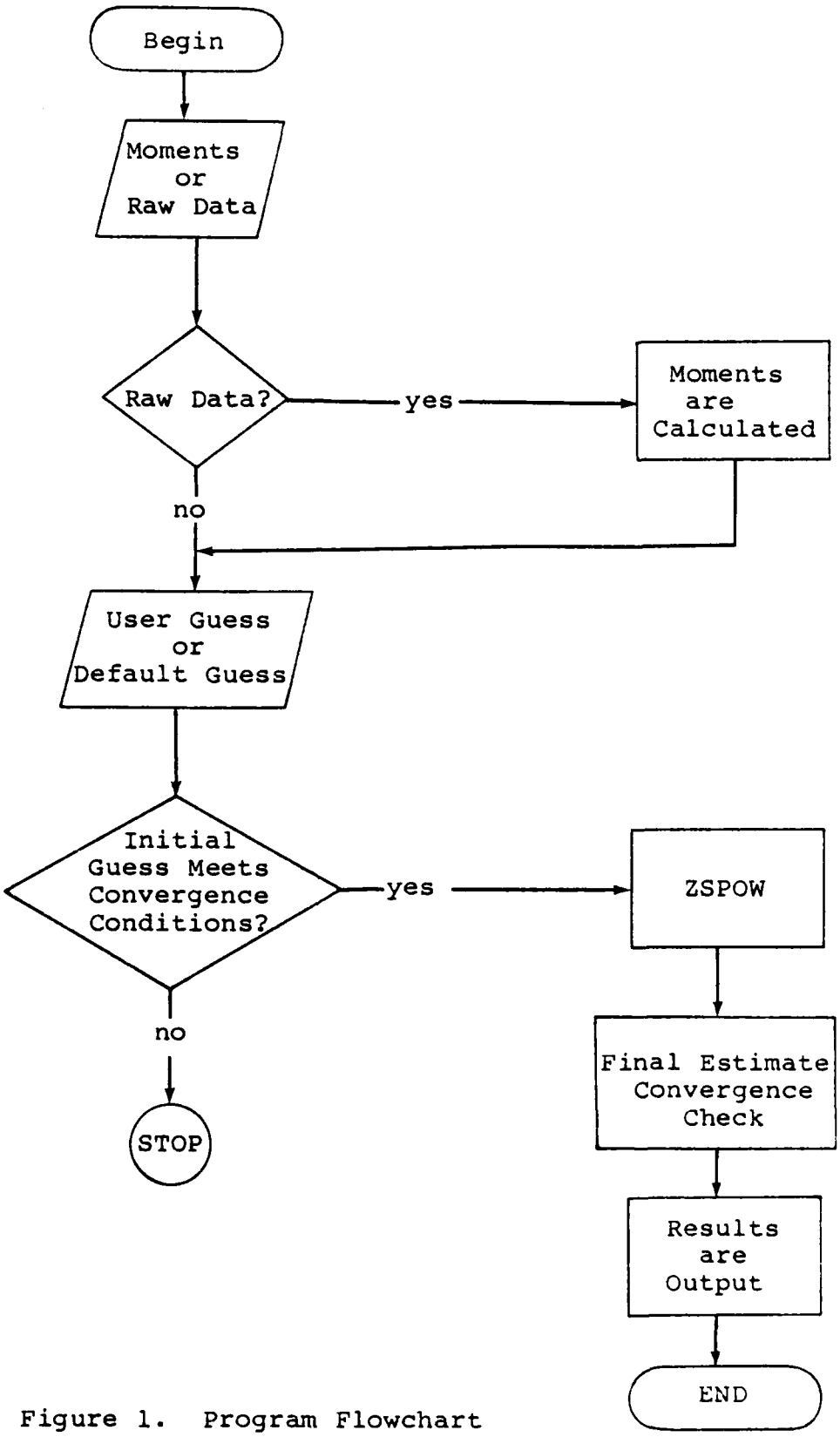

Figure 1. Program Flowchart

However, many other types of errors will cause the program to stop.

Phase Four.

The final phase added error checks and comments to the program. The comments were intended to help the user understand the program. The error messages were intended to keep the user within the bounds of the program and to give the user some idea of why the program was stopped. In order to get the most information from the error messages, it is important to know the order of error checks. For instance, if three conditions are checked and the program stops because the third condition is not satisfied, then the user can reason that the other two conditions were satisfied. The following list contains the successive order of necessary conditions for the program to continue running:

1. FLAG = 0 or 1
2. $0 \leq m \leq q$
3. $0 \leq n \leq p$
4. $p+q \leq 5$
5. TYPE = 0, 1, 2 or 3
6. If TYPE = 0, then number of moments = $2(p+q)+2$
7. If TYPE = 1, 2 or 3, then amount of data ≥ 20
8. If TYPE = 2 or 3 then Δx the same between all pairs
9. GUESS = 0 or 1
10. Initial guess meets convergence conditions
11. ZSPOW runs without IMSL errors

In addition to these error messages, the program will output an error message if the final estimate of the parameters fails to meet the convergence conditions. The difference is that the program will continue running for this last error check and output the results of ZSPOW. The reason for this has already been discussed.

Two final comments on error messages may help the program user. First of all, when the convergence conditions are not met, the program will output an error message listing the specific case that was not satisfied. These cases were listed earlier under the subheading of phase two. Second, ZSPOW outputs the following error messages in Tape 6:

1. IER=129 indicates that the maximum number of iterations has been exceeded,
2. IER=130 indicates that the desired number of significant digits is too large, and
3. IER=131 indicates that ZSPOW has not made good progress.

All the error checks were verified by making the appropriate mistake on the input tape.

The last comments added to the program consisted of the data input format for Tape 8. Although many of the terms have been defined previously throughout Chapter 5, the definitions are restated so they may be seen in one convenient location. If the zeroth moment is used, then FLAG = 0. If the moments start with the first moment, then FLAG = 1. The variable M is the number of "B" gamma functions in the numerator, N is the number of "A" gamma functions in the

numerator, P is the total number of "A" gamma functions, and Q is the total number of "B" gamma functions. If previously calculated moments are used, then TYPE = 0. If univariate data are input, then TYPE = 1. If ordered pairs from a relative frequency or from a continuous function are input, then TYPE = 2 or TYPE = 3 respectively. GUESS = 0 if the user wishes to supply his own initial guess of the H-function parameters. If the default initial guess is desired, then GUESS = 1. Finally, if TYPE = 0, then NUM is the number of moments. If TYPE = 1, then NUM is the number of univariate data points. The variable NUM is the number of ordered pairs if TYPE = 2 or TYPE = 3.

The data must be input on Tape 8 as follows:

1. FLAG, M, N, P, Q, TYPE, GUESS, NUM (Integers)
2. Data (Real) -
   - A. TYPE 0 - 2(P+Q)+2 moments from lowest to highest
   - B. TYPE 1 - NUM univariate data
   - C. TYPE 2 or 3 - NUM data pairs in the form x, f(x)
3. Initial Guess (Real) -
   - A. GUESS 0 - 2(P+Q) parameters:
     - "B" pairs in the numerator
     - "A" pairs in the numerator
     - "B" pairs in the denominator
     - "A" pairs in the denominator
   - B. GUESS 1 - No input necessary

The program was now in final form and can be seen in Appendix C. It had been verified and validated by over one hundred separate runs! We next discuss the program limitations that came about because of the decision to use the first definition of the H-function in Chapter 2 and the decision to use the method of moments in Chapter 3.

Program Limitations

One of the first decisions that was made was to use the first equation in Eq (2.1) as our definition of the H-function. This decision forced the calculation of I(s) to be done with the formula

$$I(s) = \frac{\prod_{j=1}^{m} \Gamma(b_j+B_j s) \prod_{i=1}^{n} \Gamma(1-a_i-A_i s)}{\prod_{i=n+1}^{p} \Gamma(a_i+A_i s) \prod_{j=m+1}^{q} \Gamma(1-b_j-B_j s)}$$

This formula was placed in a program subroutine named COMPIS. The program cannot solve for H-functions such as

$$H(z) = H_{p\,q}^{m\,n}\left[z^2:\{(a_i,A_i)\};\ \{(b_j,B_j)\}\right]$$

or

$$H(z) = H_{p\,q}^{m\,n}\left[\frac{1}{z}:\{(a_i,A_i)\};\ \{(b_j,B_j)\}\right]$$

because the variable z must have a power of one. This is not a serious limitation. The program user can use the

properties outlined in Chapter 2 in order to convert the H-function to the proper form of

$$H(z) = k{}^{m\ n}_{p\ q}H\left[cz:\{(a_i,A_i)\};\ \{(b_j,B_j)\}\right]$$

As an example, suppose the program user expected an H-function like

$$H(z) = {}^{0\ 1}_{1\ 0}H\left[(2z)^{-1}:(-1,1);\right]$$

He would only be able to run the program after using the reciprocal property to produce

$$H(z) = {}^{1\ 0}_{0\ 1}H\left[2z:;(2,1)\right]$$

where the variable z is taken to the first power.

The second decision to use the method of moments posed more serious problems. The first problem involved the unstable nature of the higher data moments. A sufficient discussion of this problem can be found in Chapter 3. With data moments not close to their true value, the H-function cannot be expected to accurately fit the true distribution.

The second problem with moments eliminated some functions and one distribution from our H-function curve-fitting procedure. The method of moments does not apply to those statistical distributions and functions that either do not

have defined H-function moments (μ_r) or do not have finite data moments (M_r). This included all trigometric functions, the log $(1 \pm z)$ function, and the Half-Cauchy distribution. The method of maximum likelihood estimation may allow these few remaining functions and the one distribution to be fit once the theory has progressed.

The third problem involved restrictions on three of the remaining 2^{nd} order distributions. These restrictions come from the fact that 2^{nd} order H-functions require five additional moments besides the zeroth moment. Recall that the recommended configuration for the program is FLAG = 0. In order to generate the required number of H-function moment equations, the following restrictions must be met. For a half-student distribution, the parameter Θ cannot be less than or equal to five. For an F distribution, the parameter $\emptyset$ cannot be less than or equal to ten. Finally, for a beta distribution of the second kind, the parameter $\emptyset$ cannot be less than or equal to five.

The fourth and final problem caused by the decision to use the method of moments has already been referred to as the problem of moments. Moments, in general, do not uniquely determine the distribution function when only a finite number of moments are available (29:81; 33:202-203). This can be seen in Figure 2, taken from Ramberg (33:205),

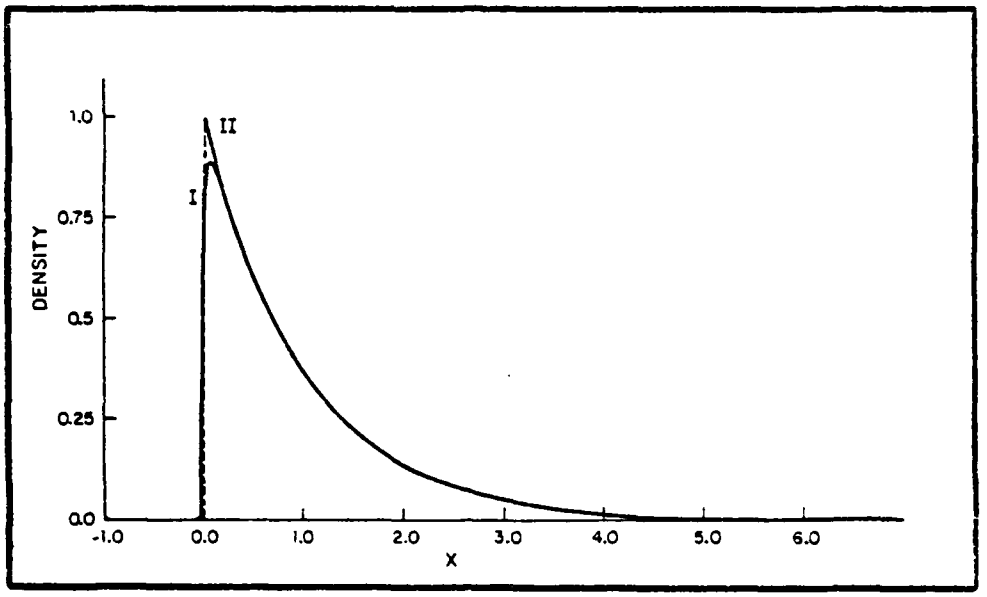

Figure 2. Two Densities With Approximately the Same First Four Moments

which compares the graphs of two different distributions that have approximately the same first four moments.

Data Generation

The next step was to create data sets from mathematical functions and statistical distributions. We decided not to use either type 1 or type 2 data. Recall that type 1 data are univariate deviates and that type 2 data are ordered pairs from a relative frequency. Some mathematical functions could not be converted into either type of data. For that matter, the conversion of univariate statistical data into a relative frequency was both time-consuming and subjective. Since graphs were desired, paired data were necessary.

Type 3 data had none of these disadvantages. Recall that type 3 data are observations of the function f(x) at various values of x. Both mathematical functions and statistical distributions could be easily converted into type 3 data. The data pairs made graphing possible. These graphs were important to provide a visual representation of the fit of the estimated H-function to the data.

The following three factors were considered when we created the type 3 data:

1. Amount of data,
2. Type of function, and
3. Order of the H-function.

The amount of data could either have a small level or a large level. For the small level, 20 data points were used. For the large level, 100 or 150 data points were used. The type of function could be either a nonstatistical, mathematical function or a statistical distribution. Examples of mathematical functions were the power and generalized gamma functions. Statistical distributions included the exponential and beta probability density functions. The third factor also had two levels, because we created both first and second order data.

As a minimum, we created 8 separate data sets to cover each factor at two levels. This demonstrated the versatility of the H-function curve-fitting procedure. In addition,

many other data sets were created so that more than one graph could be inspected.

Procedure

Once the parameter sets were generated, we used the computer program named THESIS (Appendix C) to estimate the parameters of the H-function from the raw data. These estimates could then be used in two ways.

First, if the data was from a distribution, then the estimates could be compared to Table III in order to determine the closest distribution. As an example, if THESIS returned the values

b = 6.00

B = 1.01

k = .0027

c = .5

then the program user could hypothesize that the data was from a gamma distribution. Since $\theta-1=b$ and $b=6.00$, the user would estimate $\theta=7.00$. Further, since $c=\emptyset$ and $c=.5$, the estimate of $\emptyset$ is 1/2. Therefore, the user could stop and say the data was from a gamma ($\theta=7, \emptyset=.5$). However, the user implies $B=1.00$ and $k=.0027778$ when he assumes that the data is from that specific gamma p.d.f. There is no general method to reevaluate the other parameters once the additional constraints have been added.

It is better to use the estimates of the parameters of the H-function in another way, as is done in this thesis.

TABLE III. Generalized Statistical Distributions

Distribution	H-function Distribution Parameters						
	m	n	p	q	k	c	(a,A)	(b,B)
Gamma	1	0	0	1	$\phi[\Gamma(\theta)]^{-1}$	ϕ	---	$(\theta-1,1)$
Exponential	1	0	0	1	ϕ	ϕ	---	$(0,1)$
Chi-Square	1	0	0	1	$[2\Gamma(\theta'/2)]^{-1}$	$1/2$	---	$(\frac{\theta'}{2}-1,1)$
Weibull	1	0	0	1	$\phi^{1/\theta}$	$\phi^{1/\theta}$	---	$(1-\frac{1}{\theta},\frac{1}{\theta})$
Rayleigh	1	0	0	1	$\sqrt{\phi}$	$\sqrt{\phi}$	---	$(\frac{1}{2},\frac{1}{2})$
Maxwell	1	0	0	1	$2/(\theta\sqrt{\pi})$	θ^{-1}	---	$(1,\frac{1}{2})$
Half-Normal	1	0	0	1	$[\sqrt{2\pi}\theta]^{-1}$	$[\sqrt{2}\theta]^{-1}$	---	$(0,\frac{1}{2})$
Beta (first kind)	1	0	1	1	$\Gamma(\theta+\phi)/\Gamma(\theta)$	1	$(\theta+\phi-1,1)$	$(\theta-1,1)$
Power Function	1	0	1	1	$u\theta$	1	$(u(\theta-1)+1,u)$	$(u(\theta-1),u)$
Uniform	1	0	1	1	u	1	$(1,u)$	$(0,u)$
Pareto	0	1	1	1	$u\theta$	1	$(1-u(1+\theta),u)$	$(-u(1+\theta),u)$
Half-Student	1	1	1	1	$[\sqrt{\theta\pi}\Gamma(\theta/2)]^{-1}$	$[\sqrt{\theta}]^{-1}$	$(\frac{1-\theta}{2},\frac{1}{2})$	$(0,\frac{1}{2})$
F	1	1	1	1	$\theta/[\phi\Gamma(\theta/2)\Gamma(\phi/2)]$	θ/ϕ	$(-\frac{\phi}{2},1)$	$(\frac{\theta}{2}-1,1)$
Beta (second kind)	1	1	1	1	$\phi/[\theta\Gamma(\theta)\Gamma(\phi)]$	ϕ/θ	$(-\phi,1)$	$(\theta-1,1)$

Once the estimated parameters were found, we used another computer program [7:Appendix B] to determine the value of

$$kH^{m\ n}_{p\ q}(cx)$$

for the same values of x as the data pairs. Finally, both groups of paired data were graphed together to display the fit of the estimated H-function to the actual data. These results, along with the discussion of the measure of fit between the H-function and the data, will be described in the next chapter.

VI Results

Measure of Merit

Earlier we mentioned the fact that we would use the estimated mean squared error (MSE) as our criterion for measuring the goodness-of-fit of the H-function to the data. For the i^{th} value of x, let the value for the actual function be $f(x_i)$ and the value for the estimated H-function be $H(x_i)$. Then the square of the distance between the H-function and the data point would be $[H(x_i)-f(x_i)]^2$. The sum of squares error (SSE) is then found by adding the square of the distance for all n values of x_i:

$$SSE = \sum_{i=1}^{n} [H(x_i)-f(x_i)]^2$$

Since we wanted to compare graphs with different amounts of data, the SSE needed to be adjusted for the number of data points as in

$$\text{Estimated MSE} = SSE/n$$

We expected to see a lower estimated MSE for the graphs generated from a large amount of data.

Graph Description

An invented example of the graphs is shown in Figure 3 below. This graph was not derived using the H-function curve-fitting procedure. As seen in Figure 3, a line will be used to represent the H-function data. The actual data

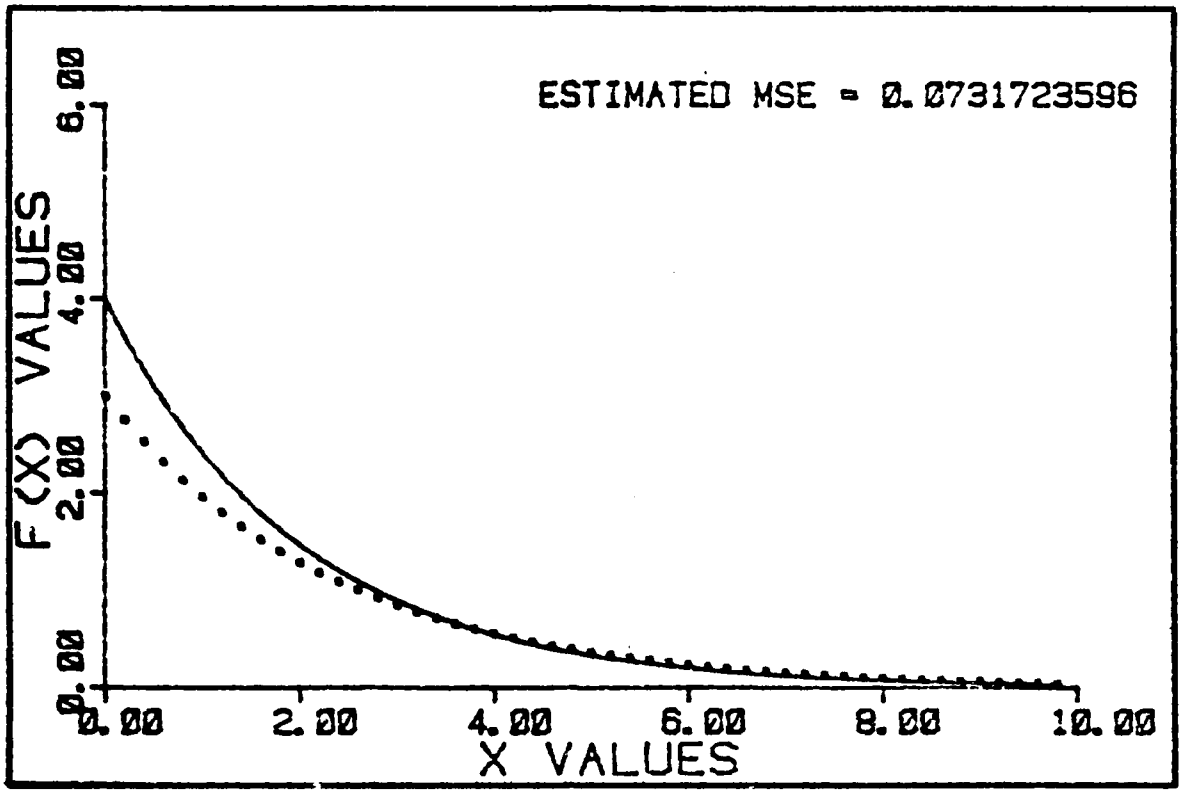

Figure 3. Sample graph

will be plotted as individual points. The estimated MSE can be found in the upper right-hand corner of the graph.

The thesis procedure used input decks like the following for a Gamma (Θ=3, $\emptyset$=2) with the amount of data equal to twenty:

```
100=0,1,0,0,1,3,0.20
110=     .2500                    .1516326649
120=     .5000                    .3678794412
130=     .7500                    .5020428603
140=    1.0000                    .5413411329
150=    1.2500                    .5130312414
160=    1.5000                    .4480836153
170=    1.7500                    .3699179469
180=    2.0000                    .2930502222
190=    2.2500                    .2249571799
200=    2.5000                    .1684486750
210=    2.7500                    .1236248360
220=    3.0000                    .0892350784
230=    3.2500                    .0635203059
240=    3.5000                    .0446822163
250=    3.7500                    .0311109958
260=    4.0000                    .0214696082
270=    4.2500                    .0147005897
280=    4.5000                    .0099961941
290=    4.7500                    .0067553776
300=    5.0000                    .0045399930
310=2.1234
320=1.2345
```

Line 100 contains FLAG, M, N, P, Q, TYPE, GUESS, and NUM. Lines 110 through 300 contain the type 3 data. The remaining two lines are the initial guess of the H-function parameters. These data decks were input into the program named THESIS which produced results such as

```
RESULTS OF ZSPOW -
 NUMERATOR:
   SMALLB(1)=      1.1816370000173736851
   BIGB(1)=       .79007616337933228 6
 DENOMINATOR:

 VALUES OF K & C ARE:
   K=      1.1122069774289684 57
   C=      1.1020198352549712 69
   FNORM=        .0000000000000000
```

FNORM is a measure of how close the estimated root vector, $\tilde{x}$, is to the true solution of the system of equations:

$$\text{FNORM} = \sum_{i=1}^{2(P+Q)} [F_i(\tilde{x})]^2$$

where F_i is the i^{th} nonlinear equation. These results were then converted into familiar H-function notation such as

$$(1.11221)\ H^{1\ O}_{O\ 1}\ [(1.10202)x:;(1.18164,.79008)]$$

Once this was done, the estimated H-function was evaluated for the same values of x as the actual data. Then both sets of paired data were graphed in Figure 4A.

Other figures for small and large amounts of data were created using the same procedure. They can be found in the following lists:

First Order Statistical		Figures
1.	Gamma (θ=3,$\emptyset$=2)	4 A/B
2.	Exponential ($\emptyset$=1/2)	5 A/B
3.	Chi-Square (θ'=4)	6 A/B
4.	Weibull (θ=3/2,$\emptyset$=1)	7 A/B
5.	Rayleigh ($\emptyset$=4)	8 A/B
6.	Maxwell (θ=2)	9 A/B
7.	Half-Normal (θ=1)	10 A/B

Second Order Statistical	Figures
1. Beta (Θ=2,$\emptyset$=3)	11 A/B
2. Power Function (Θ=3)	12 A/B
3. Uniform	13 A/B
4. Half-Student (Θ=16)	14 A/B
5. Bessel (Θ=1,$\emptyset$=1)	15

First Order Functional	Figures
1. Generalized Gamma Function (b=1,B=1/2)	16 A/B
2. Generalized Gamma Function (b=1,B=1)	17 A/B

Second Order Functional	Figures
1. z^b (b=2)	18 A/B
2. z^b (b=1)	19 A/B
3. z^b (b=-1)	20 A/B
4. $z^b(1-z)^{+a}$ (b=1/2,a=1)	21 A/B
5. $z^b(1-z)^{+a}$ (b=1/2,a=2)	22 A/B

The figure number can be used to reference the correct graph. The graphs were kept together and placed in Appendix D for quick comparison.

In addition to Appendix D, Table IV summarizes the estimated MSE for each figure. The low estimated MSE for each graph demonstrates the ability of the H-function to fit various sets of raw data. Another reassuring finding is that the H-function fits better when more data is available.

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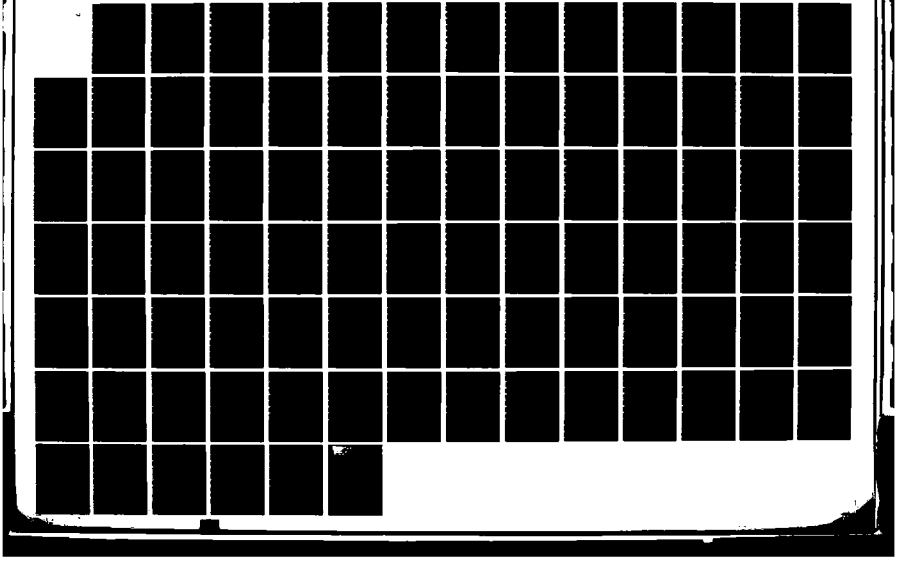

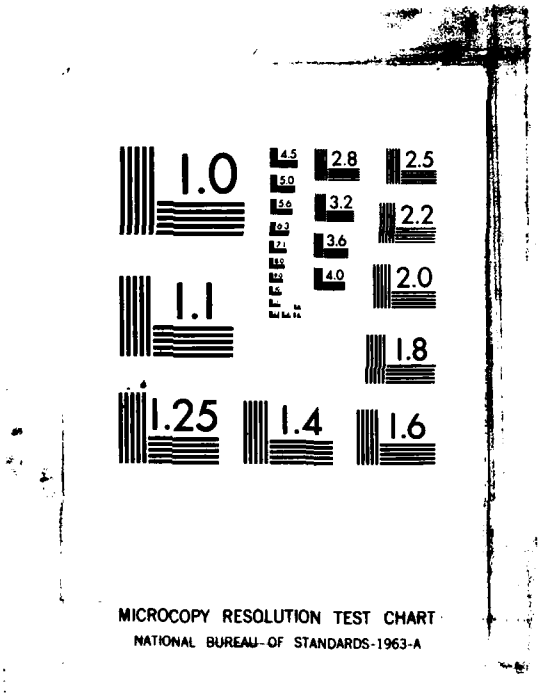

TABLE IV

Estimated Mean Squared Errors

Figure	AMOUNT OF DATA	
	Small	Large
4 A/B	0.0000658751	0.0000003710
5 A/B	0.0003411134	0.0000588732
6 A/B	0.0000153797	0.0000001745
7 A/B	0.0005039677	0.0000054808
8 A/B	0.0001306474	0.0000000673
9 A/B	0.0000007455	0.0000000006
10 A/B	0.0021639213	0.0001184473
11 A/B	0.0018068877	0.0000004352
12 A/B	0.0027320795	0.0000056500
13 A/B	0.0052858926	0.0016352035
14 A/B	0.0014194577	0.0002971406
15	---	0.0000537173
16 A/B	0.0001038111	0.0000000003
17 A/B	0.0000673331	0.0000006979
18 A/B	0.0002744282	0.0000006392
19 A/B	0.0006307255	0.0000010364
20 A/B	0.6901255958	2.0283752311
21 A/B	0.0029725091	0.0000013842
22 A/B	0.0029537131	0.0000013089

This can be seen when the estimated MSE values for large amounts of data are lower.

Demonstration Runs

The list above covers many statistical distributions and mathematical functions. In fact, all first order statistical distributions listed in Chapter 2 were run (Figures 4-10). For second order statistical distributions, those without restrictions on the distribution parameters were run (Figures 11-13, 15). In addition, the second order half-student distribution was run as a first order distribution to analyze the effect of underestimating the true order (Figure 14). Finally, a group of first and second order mathematical functions with various shapes was run (Figures 16-22).

All the runs above were done with type 3 raw data. The program could also be used with previously calculated perfect moments to demonstrate the special properties of the H-function discussed in Chapter 2. The input decks differed slightly because the type 3 data was replaced by a single line of perfect moments.

The reduction properties were demonstrated as follows. Perfect moments were derived for an exponential ($\emptyset=1/2$). For the first reduction method, an "A" gamma function in the denominator had to have the same parameters as a "B" gamma function in the numerator. When these two cancelled, an additional correct "B" gamma function had to be left in the

numerator. The input deck for the $kH^{2\ O}_{1\ 2}[cx]$ looked like

```
100=0,2,0,1,2,0,0,8
110=1,2,8,48,384,3840,46080,645120
120=.342
130=1.316
140=2.222
150=.106
160=2.987
170=.543
```

The results from THESIS were

```
RESULTS OF ZSPOW -
 NUMERATOR:
   SMALLB(1)=      -.0000001075344861 33
   BIGB(1)=       .9999998635161162 50
   SMALLB(2)=      2.7721537711625359 18
   BIGB(2)=       .8567118870611416 53
 DENOMINATOR:
   SMALLA(1)=      2.7721537300346597 02
   BIGA(1)=       .8567117533244683 61

 VALUES OF K & C ARE:
   K=       .4999998215102827 71
   C=       .4999999774120080 79
   FNORM=        .0000000000000008 99
```

Note the nearly equivalent parameters in the numerator and the denominator. When reduced, the second order H-function approximated the true first order H-function of

$$\frac{1}{2}\,H^{1\ O}_{O\ 1}\left[\frac{1}{2}x;;(O,1)\right].$$

Therefore, the first reduction method worked.

The second reduction method works when an A_i or B_j vanishes. For the same exponential distribution, the input deck for kH_1^0 [cx] $_1^1$ was

```
100=0,1,0,1,1,0,0,6
110=1,2,8,48,384,3840
120=.342
130=1.316
140=2.222
150=.106
```

The results from THESIS were

```
RESULTS OF ZSPOW -
 NUMERATOR:
   SMALLB(1)=       .000000000070760199
   BIGB(1)=      1.000000000109487530
 DENOMINATOR:
   SMALLA(1)=      4.035573643344662287
   BIGA(1)=       .000011372100772754

 VALUES OF K & C ARE:
   K=      3.137655798202459323
   C=       .499992800657203418
   FNORM=        .000000000000000000
```

Note the A_1 in the denominator is approaching zero. Therefore, the "A" gamma function was essentially a constant $\Gamma(4.036)$ term in the denominator. This term was evaluated to be 6.279. When brought out of the integral and under the k, the true k became 3.138 divided by 6.279 or 0.4998. This approximated the true first order H-function so the second reduction method also worked.

Next, the generalization of the uniform distribution

was demonstrated. The input decks for $kH_{1\,1}^{1\,0}[cx]$ were

```
100=0,1,0,1,1,0,0,6
110=1,.5,.33333333,.25,.2,.16666667
120=.14567
130=.96543
140=1.23456
150=.916432
```

and

```
100=0,1,0,1,1,0,0,6
110=1,.5,.33333333,.25,.2,.16666667
120=.675
130=2.111
140=.337
150=1.998
```

Note that only the initial guess of the H-function parameters was changed. The first initial guess gave results from THESIS as follows:

```
RESULTS OF ZSPOW -
 NUMERATOR:
   SMALLB(1)=        .000001122240934309
   BIGB(1)=       1.00864174078524911
 DENOMINATOR:
   SMALLA(1)=       1.000006384974028606
   BIGA(1)=       1.00886500279229O966

 VALUES OF K & C ARE:
   K=       1.008864845751389794
   C=        .9999968377389265S8
   FNORM=         .000000000000015405
```

These results were close to the nongeneralized uniform

distribution $H^{1\ 0}_{1\ 1}[x:(1,1);\ (0,1)]$.

When the initial guess was changed, the results became

```
RESULTS OF ZSPOW -
 NUMERATOR:
   SMALLB(1)=       .000082785039857461
   BIGB(1)=     15.426639569366727756
 DENOMINATOR:
   SMALLA(1)=      1.000092670887106294
   BIGA(1)=     15.426641384669039780

 VALUES OF K & C ARE:
   K=     15.427045040837867873
   C=        .999988515678520429
   FNORM=        .000000000000008917
```

These results demonstrated the generalized uniform distribu-

tion $uH^{1\ 0}_{1\ 1}[x:(1,u);\ (0,u)]$ where u=15.427. Looking back at the first uniform, this also was generalized but u=1.009.

When these two specific H-functions were evaluated for x values, the following results were obtained:

u=1.009		u=15.427	
.0500	.999999	.0500	.999999
.1000	1.000000	.1000	1.000000
.1500	1.000000	.1500	1.000000
.2000	1.000000	.2000	1.000000
.2500	1.000000	.2500	1.000000
.3000	.999999	.3000	1.000000
.3500	.999999	.3500	.999999
.4000	.999999	.4000	.999999
.4500	.999998	.4500	.999998
.5000	.999998	.5000	.999997
.5500	.999998	.5500	.999996
.6000	.999997	.6000	.999995
.6500	.999996	.6500	.999994
.7000	.999995	.7000	.999992
.7500	.999995	.7500	.999991
.8000	.999993	.8000	.999989
.8500	.999991	.8500	.999987
.9000	.999989	.9000	.999985
.9500	.999984	.9500	.999982
1.0000	.999959	1.0000	.999979

Both H-functions were a good approximation to the uniform distribution.

The generalization of the Pareto distribution (θ=6) was also demonstrated. The input deck for ${}_{1}^{0}kH_{1}^{1}[cx]$ was

```
100=0,0,1,1,1,0,0,6
110=1,1.2,1.5,2,3,6
120=-5.654
130=1.112
140=-7.174
150=.987
```

This gave results from THESIS of

```
RESULTS OF ZSPOW -
 NUMERATOR:
   SMALLA(1)=     -3.832789655700494791
   BIGA(1)=       .690398445680816764
 DENOMINATOR:
   SMALLB(1)=     -4.832794475434639025
   BIGB(1)=       .690398684011771735

 VALUES OF K & C ARE:
   K=      4.142424973174115621
   C=      1.000001072468101881
   FNORM=        .000000000000014109
```

These results demonstrated the generalized Pareto distribu-

tion $u\theta H_{1\,1}^{0\,1}[x:(1-u(1+\theta),u);(-u(1+\theta),u)]$ where u=0.6904.

The H-function was then evaluated for x values. These data pairs were then compared to an actual Pareto distribution with θ=6:

1.2500	1.250287	1.2500	1.258291
1.5000	.351166	1.5000	.351166
1.7500	.119367	1.7500	.119367
2.0000	.046875	2.0000	.046875
2.2500	.020553	2.2500	.020553
2.5000	.009830	2.5000	.009830
2.7500	.005045	2.7500	.005045
3.0000	.002743	3.0000	.002743
3.2500	.001567	3.2500	.001567
3.5000	.000933	3.5000	.000933
3.7500	.000575	3.7500	.000575
4.0000	.000366	4.0000	.000366
4.2500	.000240	4.2500	.000240
4.5000	.000161	4.5000	.000161
4.7500	.000110	4.7500	.000110
5.0000	.000077	5.0000	.000077
5.2500	.000055	5.2500	.000055
5.5000	.000039	5.5000	.000039
5.7500	.000029	5.7500	.000029
6.0000	.000021	6.0000	.000021

The estimated H-function produced a good fit to the actual distribution.

Now that some properties have been demonstrated, the usefulness of the program is apparent. One further run may emphasize this point. While attempting to produce an example for the reduction property, the following input data deck was used

```
100=0,1,0,1,1,0,0,6
110=1,2,8,48,384,3840
120=.342
130=1.316
140=2.222
150=.886
```

The results of THESIS were

```
RESULTS OF ZSPOW -
 NUMERATOR:
   SMALLB(1)=      -.000000002439904119
   BIGB(1)=      2.000000030682542729
 DENOMINATOR:
   SMALLA(1)=       .499999998705995097
   BIGA(1)=      1.000000030809658824

 VALUES OF K & C ARE:
   K=       1.772453868441800751
   C=       2.000000041492725700
   FNORM=        .000000000000000000
```

We were expecting to see an A_1 near zero. Instead the program produced $1.772\ H^{1\ 0}_{1\ 1}[2x;(\frac{1}{2},1);\ (0,2)]$. But the perfect moments were from $\frac{1}{2}\ H^{1\ 0}_{0\ 1}[\frac{1}{2}x;;(0,1)]$. At first glance, the two H-functions did not appear to be equivalent.

However, using rule 6.1.18 from Abramowitz and Stegun (2:256) and the fact that $k=\sqrt{\pi}$, the two H-functions were shown to be equal:

$$\sqrt{\pi}\, H^{1\,0}_{1\,1}\left[2x:\left(\tfrac{1}{2},1\right);\ (0,2)\right]$$

$$= \sqrt{\pi}\,\frac{1}{2\pi i}\int_c \frac{\Gamma(2s)}{\Gamma(\frac{1}{2}+s)}\,(2x)^{-s}\,ds$$

$$= \frac{\sqrt{\pi}}{2\pi i}\int_c \frac{2^{2s}\Gamma(s)\,\Gamma(\frac{1}{2}+s)}{\sqrt{2}\,\sqrt{2\pi}\,\Gamma(\frac{1}{2}+s)}\,(2x)^{-s}\,ds$$

$$= \frac{1}{2\pi i}\int_c (2^{s-1})\Gamma(s)\,x^{-s}\,ds$$

$$= \frac{1}{2}\,\frac{1}{2\pi i}\int_c \Gamma(s)\,(\tfrac{1}{2}x)^{-s}\,ds$$

$$= \frac{1}{2}\,H^{1\,0}_{0\,1}\left[\tfrac{1}{2}x:(0,1)\right]$$

In a less theoretical approach, the H-function was simply evaluated for values of x. The results that follow correspond exactly to an exponential distribution with $\emptyset=1/2$:

.7500	.343645
1.5000	.236183
2.2500	.162326
3.0000	.111565
3.7500	.076677
4.5000	.052700
5.2500	.036220
6.0000	.024894
6.7500	.017109
7.5000	.011759
8.2500	.008082
9.0000	.005554
9.7500	.003818
10.5000	.002624
11.2500	.001803
12.0000	.001239
12.7500	.000852
13.5000	.000585
14.2500	.000402
15.0000	.000277

This concludes the discussion about the results of the thesis. More detailed conclusions about the efficiency and effectiveness of the H-function curve-fitting procedure follow in the next chapter. Also, new findings are highlighted and further studies are recommended.

VII Conclusions and Remarks

Summary

We have developed a procedure to estimate the parameters of the H-function which gives the best fit to a set of raw data. The procedure uses the method of moments and can be used with both mathematical functions and continuous statistical distributions defined over positive x. Our computer program will accept univariate data, data pairs, or moments previously calculated from data. The user has the option of using the zeroth moment or beginning with the first moment. The user can supply his own initial guess of the parameters or allow the program to use a default initial guess. The program automatically checks the initial guess and final estimate of the H-function parameters against the convergence conditions of the H-function. If the program should stop before completion, it also has other diagnostic checks built in which will give the user some indication of the error which caused the program to abort. If no errors are found, the program will output the parameters of the fitted H-function.

The method of moments does not always produce the "best" estimates of a distribution's parameters. High moments calculated from data tend to be inaccurate. Further, moments do not uniquely define a distribution. Still, our experience has shown the method of moments to be an effective way to estimate the parameters of the H-function.

Because the analytic moments of an H-function are easily derived using the Mellin transformation, the equations of the method of moments can be simply written.

We used an IMSL routine named ZSPOW to solve these nonlinear equations for the unknown parameters. ZSPOW contains Powell's quasi-Newton hybrid algorithm for systems of nonlinear equations. This method requires a reasonably close initial guess and does not guarantee convergence but these restrictions are common to most techniques. Powell's method also provides super linear convergence.

The estimated H-function parameters can be adjusted using Table III if a named statistical distribution was desired. Alternatively, they could be used as inputs to another computer program (7:Appendix B) which would calculate the H-function at certain values of x and plot the probability density function (p.d.f.) and cumulative distribution function (c.d.f.).

We tested our procedure using many mathematical functions and statistical distributions. The results were impressive and are presented in Table IV and Appendix D.

Since many mathematical functions and statistical distributions are simultaneously considered when the H-function is fit to a set of data, fewer separate tests are required. This generalization alone will increase the efficiency of curve-fitting and density estimation.

Another benefit of simultaneously considering many functions and distributions is that there is a higher probability of finding the function or distribution which "best" fits the data. Thus, effectiveness increases when the generalized method is used.

New Findings

We made several contributions to the theory of H-functions. We discovered a new reduction property for H-functions. We corrected typographical errors in Mathai and Saxena (26) for arcsin(z), arctanh(z), and $\log(1\pm z)$. We gave generalized H-function formulas for the logarithmic function log(z) and power function z^b. We showed that the Pareto p.d.f. can be expressed as an H-function. Finally, we generalized the H-function formulas for the Power Function p.d.f., the Uniform p.d.f., and the Pareto p.d.f. These new results were presented in Chapter 2 and the details that are not obvious are presented in Appendix A.

Recommendations for Further Research

Because the c.d.f. of a random variable with an H-function distribution is simply one minus another H-function, this constraint could be added when fitting a density. Fitting both the p.d.f. and c.d.f. could enhance the power and versatility of the H-function method.

Further research is also needed to develop maximum likelihood estimates for the H-function parameters. The

derivation was started and can be seen in Chapter 3. Our results need to be extended.

Modifications to our program could also be made to allow certain parameters to be fixed throughout the IMSL routine ZSPOW. For example, the user might find that after the first run is made, B=.97. He might want to fix B=1 and solve for the other parameters, given this constraint. We could not provide this option easily because ZSPOW does not allow a variable to be altered within the subroutine that defines the system of nonlinear equations.

APPENDIX A

Outlines of Proofs of New Findings

The purpose of this appendix is to outline proofs which verify our new theoretical results. Although these proofs are not central to the purpose of the thesis, we felt compelled to verify our new findings so that others might further extend the H-function theory.

Throughout this appendix it is assumed that the reader thoroughly understands the process of evaluating contour integrals in the complex plane by summing the residues at the singularities (or poles) of the integrand.

Corrected Mathematical Functions

$$\arcsin(z) = -\frac{i}{4\sqrt{\pi}}\, H^{1\,2}_{2\,2}\left[iz:(1,\tfrac{1}{2}),(1,\tfrac{1}{2});\ (\tfrac{1}{2},\tfrac{1}{2}),\ (0,\tfrac{1}{2})\right]$$

$$D = 1 \quad E = 0 \quad L = -\frac{3}{2} \quad R = 1$$

type VI $\quad +\sum$ LHP residues for $|z| \leq 1$

$\quad -\sum$ RHP residues for $|z| > 1$

$$H^{1\,2}_{\ 2}\left[iz:(1,\tfrac{1}{2}),\ (1,\tfrac{1}{2});\ (\tfrac{1}{2},\tfrac{1}{2}),\ (0,\tfrac{1}{2})\right]$$

$$\frac{1}{2\pi i}\int_c \frac{\Gamma(\frac{1}{2}+\frac{1}{2}s)[\Gamma(-\frac{1}{2}s)]^2}{\Gamma(1-\frac{1}{2}s)}\,(iz)^{-s}\,ds$$

numerator poles can be separated

by any w in the open interval (-1,0)

$\Gamma(\frac{1}{2}+\frac{1}{2}s)$ has poles of order 1 at

$s_J=-(2J+1)$, $J=0,1,2,\ldots$

$$f_J(s) = \frac{(s-s_J)\Gamma(\frac{1}{2}+\frac{1}{2}s)[\Gamma(-\frac{1}{2}s)]^2}{\Gamma(1-\frac{1}{2}s)}\,(iz)^{-s}$$

$$= \frac{(s+2J+1)\Gamma(\frac{1}{2}+\frac{1}{2}s+J+1)[\Gamma(-\frac{1}{2}s)]^2(iz)^{-s}}{(\frac{1}{2}+\frac{1}{2}s)(\frac{1}{2}+\frac{1}{2}s+1)\cdots(\frac{1}{2}+\frac{1}{2}s+J)(-\frac{1}{2}s)\Gamma(-\frac{1}{2}s)}$$

$$= \frac{2\Gamma(\frac{1}{2}+\frac{1}{2}s+J+1)\Gamma(-\frac{1}{2}s)(iz)^{-s}}{(\frac{1}{2}+\frac{1}{2}s)\cdots(\frac{1}{2}+\frac{1}{2}s+J-1)(-\frac{1}{2}s)}$$

$$f_J(s_J) = \frac{2\Gamma(J+\frac{1}{2})(iz)^{2J+1}}{(-1)^J\ J!\ (J+\frac{1}{2})}$$

$$= \frac{2i\Gamma(J+\frac{1}{2})z^{2J+1}}{J!\ (J+\frac{1}{2})}$$

$$\arcsin(z) = -\frac{i}{4\sqrt{\pi}}\,\frac{1}{2\pi i}\left\{\ 2\pi i \sum_{J=0}^{\infty}\frac{2i\Gamma(J+\frac{1}{2})z^{2J+1}}{J!\ (J+\frac{1}{2})}\ \right\}$$

$$= -\frac{i}{4\sqrt{\pi}}\sum_{J=0}^{\infty} 2i\frac{(1.3.5\ldots(2J+1))\sqrt{\pi}}{2^{J+1}\ (J+\frac{1}{2})}\,\frac{z^{2J+1}}{J!(J+\frac{1}{2})}$$

$$= \frac{1}{4}\sum_{J=0}^{\infty}\frac{1.3.5\ldots(2J+1)z^{2J+1}}{2^J\ J!\,(J+\frac{1}{2})^2}$$

$$= z + \frac{1}{2.3}z^3 + \frac{1.3}{2.4.5}z^5 + \frac{1.3.5}{2.4.6.7}z^7 + \ldots$$

which is the infinite series for arcsin(z). The proof is similar for $|z| > 1$.

$$\operatorname{arctanh}(z) = -\frac{i}{4}\, H^{1\,2}_{2\,2}\left[iz:(1,\tfrac{1}{2}),\ (\tfrac{1}{2},\tfrac{1}{2});\ (\tfrac{1}{2},\tfrac{1}{2}),\ (0,\tfrac{1}{2})\right]$$

$D = 1 \quad E = 0 \quad L = -1 \quad R = 1$

type VI $\quad +\sum$ LHP residues for $|z| < 1$

$\quad -\sum$ RHP residues for $|z| > 1$

$$H^{1\,2}_{2\,2}\left[iz:(1,\tfrac{1}{2}),\ (\tfrac{1}{2},\tfrac{1}{2});\ (\tfrac{1}{2},\tfrac{1}{2}),\ (0,\tfrac{1}{2})\right]$$

$$= \frac{1}{2\pi i}\int_c \frac{\Gamma(\frac{1}{2}+\frac{1}{2}s)\Gamma(-\frac{1}{2}s)\Gamma(\frac{1}{2}-\frac{1}{2}s)}{\Gamma(1-\frac{1}{2}s)}\,(iz)^{-s}\,ds$$

numerator poles can be separated

by any w in the open interval (-1,0)

$\Gamma(\frac{1}{2}+\frac{1}{2}s)$ has poles of order 1 at

$s_J = -(2J+1)$, $J = 0,1,2\ldots$

$$f_J(s) = \frac{2\Gamma(\frac{1}{2}+\frac{1}{2}s+J+1)\Gamma(\frac{1}{2}-\frac{1}{2}s)(iz)^{-s}}{(\frac{1}{2}+\frac{1}{2}s)\ldots(\frac{1}{2}+\frac{1}{2}s+J-1)(-\frac{1}{2}s)}$$

$$f_J(s_J) = \frac{2\Gamma(J+1)(iz)^{2J+1}}{(-1)^J\, J!\ (J+\frac{1}{2})}$$

$$= \frac{4i\ z^{2J+1}}{2J+1}$$

$$\operatorname{arctanh}(z) = -\frac{i}{4}\,\frac{1}{2\pi i}\left\{ 2\pi i \sum_{j=0}^{\infty} \frac{4i\ z^{2J+1}}{2J+1} \right\}$$

$$= \sum_{J=0}^{\infty} \frac{z^{2J+1}}{2J+1}$$

$$= z + \frac{z^3}{3} + \frac{z^5}{5} + \ldots$$

which is the infinite series for arctanh(z). The proof is similar for $|z| > 1$.

$$\log(1\pm z) = H^{1\ 2}_{2\ 2}[\pm z:(1,1),\ (1,1);\ (1,1),\ (0,1)]$$

$D = 2 \quad E = 0 \quad L = -1 \quad R = 1$

type VI $\quad +\sum$ LHP residues for $|z| < 1$

$\quad -\sum$ RHP residues for $|z| > 1$

$$H^{1\ 2}_{2\ 2}[\pm z:(1,1),\ (1,1);\ (1,1),\ (0,1)]$$

$$= \frac{1}{2\pi i}\int_c \frac{\Gamma(1+s)[\Gamma(-s)]^2}{\Gamma(1-s)}\ (\pm z)^{-s}\ ds$$

numerator poles can be separated by any w in the open interval $(-1,0)$

$\Gamma(1+s)$ has poles of order 1 at $s_J=-(J+1)$, $J=0,1,2\ldots$

$$f_J(s) = \frac{\Gamma(1+s+J+1)[\Gamma(-s)]^2(\pm z)^{-s}}{(1+s)(1+s+1)\ldots(1+s+J-1)\Gamma(1-s)}$$

$$= \frac{\Gamma(1+s+J+1)\Gamma(-s)(\pm z)^{-s}}{(1+s)(1+s+1)\ldots(1+s+J-1)(-s)}$$

$$f_J(s_J) = \frac{\Gamma(J+1)(\pm z)^{J+1}}{(-1)^J J!(J+1)}$$

$$= \frac{(\pm z)^{J+1}}{(-1)^J(J+1)}$$

$$\log(1\pm z) = \frac{1}{2\pi i}\left\{ 2\pi i \sum_{J=0}^{\infty} \frac{(\pm z)^{J+1}}{(-1)^J (J+1)} \right\}$$

$$= \sum_{J=0}^{\infty} \frac{(\pm z)^{J+1}}{(-1)^J (J+1)}$$

$$\log(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \ldots$$

$$\log(1-z) = -z - \frac{z^2}{2} - \frac{z^3}{3} - \frac{z^4}{4} - \ldots$$

which are the infinite series for log(1+z) and log(1-z). The proof is similar for $|z| > 1$.

Generalized Mathematical Functions

$$\log(z) = -u^2 \, H^{2\,0}_{2\,2}[z:(1,u),\ (1,u);\ (0,u),\ (0,u)]$$

$$0 < z \leq 1$$

$D = 0 \quad E = 0 \quad L = -2 \quad R = 1$

type VI $\quad +\sum$ LHP residues for $|z| \leq 1$

no RHP singularities

$$\log(z) = \frac{-u^2}{2\pi i} \int_c \frac{[\Gamma(us)]^2}{[\Gamma(1+us)]^2} z^{-s}\, ds$$

$$= \frac{-u^2}{2\pi i} \int_c \frac{1}{(us)^2} z^{-s}\, ds$$

$$= \frac{-1}{2\pi i} \int_c \frac{1}{s^2} z^{-s}\, ds$$

$$= \frac{-1}{2\pi i} \int_c \frac{[\Gamma(s)]^2}{[\Gamma(1+s)]^2} z^{-s}\, ds$$

$$= -H^{2\ 0}_{2\ 2}[z:(1,1),\ (1,1);\ (0,1),\ (0,1)]$$

$$0 < z \leq 1$$

Q. E. D.

$$\log(z) = u^2\, H^{0\ 2}_{2\ 2}[z:(1,u),\ (1,u);\ (0,u),\ (0,u)]$$

$D = 0 \quad E = 0 \quad L = -2 \quad R = 1$

type VI no LHP singularities

$-\sum$ RHP singularities for $|z| > 1$

$$\log(z) = \frac{u^2}{2\pi i}\int_c \frac{[\Gamma(-us)]^2}{[\Gamma(1-us)]^2}\, z^{-s}\, ds$$

$$= \frac{u^2}{2\pi i}\int_c \frac{1}{(-us)^2}\, z^{-s}\, ds$$

$$= \frac{1}{2\pi i}\int_c \frac{[\Gamma(-s)]^2}{[\Gamma(1-s)]^2}\, z^{-s}\, ds$$

$$= H^{0\ 2}_{2\ 2}[z:(1,1),\ (1,1);\ (0,1),\ (0,1)]$$

$$z > 1$$

Q. E. D.

Statistical Distributions

Power Function p.d.f.

$$f(x|\theta) = \theta x^{\theta-1} \qquad 0 < x < 1$$

$$\theta > 0$$

$$= u\theta H^{1\ 0}_{1\ 1}[x:(u(\theta-1)+1,u);\ (u(\theta-1),u)]$$

$D = 0 \quad E = 0 \quad L = -1 \quad R = 1$

type VI $+\sum$ LHP residues for $|x| < 1$

no RHP singularities

$$f(x|\theta) = \frac{u\theta}{2\pi i}\int_C \frac{\Gamma(u(\theta-1)+us)\ x^{-s}}{\Gamma(u(\theta-1)+1+us)}\,ds$$

$$= \frac{u\theta}{2\pi i}\int_C \frac{1}{(u(\theta-1)+us)}\ x^{-s}\,ds$$

$$= \frac{\theta}{2\pi i}\int_C \frac{1}{\theta-1+s}\ x^{-s}\,ds$$

$$= \frac{\theta}{2\pi i}\{2\pi i\ x^{-(1-\theta)}\}$$

$$= \theta x^{\theta-1} \qquad \text{Q. E. D.}$$

Note: This generalization also applies to the mathematical power function z^b.

Uniform p.d.f.

$$f(x) = 1 \qquad 0 < x < 1$$

$$= uH^{1\ 0}_{1\ 1}[x:(1,u);\ (0,u)]$$

$D = 0 \quad E = 0 \quad L = -1 \quad R = 1$

type VI $+\sum$ LHP residues for $|x| < 1$

no RHP singularities

$$f(x) = \frac{u}{2\pi i}\int_C \frac{\Gamma(us)\ x^{-s}}{\Gamma(1+us)}\,ds$$

$$= \frac{u}{2\pi i}\int_C \frac{1}{us}\ x^{-s}\,ds$$

$$= \frac{1}{2\pi i}\int_C \frac{1}{s}\ x^{-s}\,ds$$

$$= \frac{1}{2\pi i}\{2\pi i\ x^{0}\} = 1 \qquad \text{Q. E. D.}$$

Pareto p.d.f.

$$f(x|\theta) = \theta x^{-\theta-1} \qquad x > 1, \quad \theta > 0$$

$$= \theta H^{0\ 1}_{1\ 1}[x:(-\theta,1);\ (-\theta-1,1)]$$

$D = 0 \quad E = 0 \quad L = -1 \quad R = 1$

type VI no LHP singularities

$-\sum$ RHP residues for $|x| > 1$

$$f(x|\theta) = \frac{\theta}{2\pi i}\int_c \frac{\Gamma(1+\theta-s)}{\Gamma(2+\theta-s)}\, x^{-s}\, ds$$

$$= \frac{\theta}{2\pi i}\int_c \frac{1}{1+\theta-s}\, x^{-s}\, ds$$

$$= \frac{\theta}{2\pi i}\{2\pi i\ x^{-(1+\theta)}\}$$

$$= \theta x^{-\theta-1} \qquad \text{Q. E. D.}$$

Pareto p.d.f. (generalized)

$$f(x|\theta) = \theta x^{-\theta-1} \qquad x > 1, \quad \theta > 0$$

$$= u\theta H^{0\ 1}_{1\ 1}[x:(1-u(1+\theta),u);\ (-u(1+\theta),u)]$$

$D = 0 \quad E = 0 \quad L = -1 \quad R = 1$

type VI no LHP singularities

$-\sum$ RHP residues for $|x| > 1$

$$f(x|\theta) = \frac{u\theta}{2\pi i} \int_C \frac{\Gamma(u(1+\theta)-us)}{\Gamma(1+u(1+\theta)-us)} x^{-s}\, ds$$

$$= \frac{u\theta}{2\pi i} \int_C \frac{1}{u(1+\theta)-us} x^{-s}\, ds$$

$$= \frac{\theta}{2\pi i} \int_C \frac{1}{1+\theta-s} x^{-s}\, ds$$

$$= \frac{\theta}{2\pi i} \{2\pi i\, x^{-(1+\theta)}\}$$

$$= \theta x^{-\theta-1}$$ Q. E. D.

APPENDIX B

Advanced Mathematical Functions and
Statistical Distributions Expressed as H-functions

The H-function also includes as special cases several advanced mathematical functions and statistical distributions (26:10-12,145-159; 37:205-206; 7:41-43,87,93).

Mathematical Functions

Bessel Functions:

$$J_v(z) = \frac{1}{2} H^{1\,0}_{0\,2}\left[\frac{z}{2}:\left(\frac{v}{2},\frac{1}{2}\right),\ \left(-\frac{v}{2},\frac{1}{2}\right)\right]$$

$$K_v(z) = \frac{1}{4} H^{2\,0}_{0\,2}\left[\frac{z}{2}:\left(\frac{v}{2},\frac{1}{2}\right),\ \left(-\frac{v}{2},\frac{1}{2}\right)\right]$$

$$Y_v(z) = \frac{1}{2} H^{2\,0}_{1\,3}\left[\frac{z}{2}:\left(-\frac{v+1}{2},\frac{1}{2}\right);\ \left(\frac{v}{2},\frac{1}{2}\right),\ \left(-\frac{v}{2},\frac{1}{2}\right),\ \left(-\frac{v+1}{2},\frac{1}{2}\right)\right]$$

$$J^{u}_{v}(z) = H^{1\,0}_{0\,2}\left[z:(0,1),\ (-v,u)\right]$$

(Maitland's generalized Bessel function)

Hypergeometric Functions:

$$M(a,b,-z) = {}_1F_1\ (a;b;-z)$$

$$= \frac{\Gamma(b)}{\Gamma(a)} H^{1\,1}_{1\,2}\left[z:(1-a,1);\ (0,1),\ (1-b,1)\right]$$

(Confluent Hypergeometric function)

$$ {}_2F_1\,(a,b;c;-z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)}\, H^{1\ 2}_{2\ 2}[z:(1-a,1),$$

$$(1-b,1);\ (0,1),\ (1-c,1)]$$

(Hypergeometric function)

$$ {}_pF_q\,(\{a_i\};\{b_j\};-z) = \frac{\prod_{j=1}^{q}\Gamma(b_j)}{\prod_{i=1}^{p}\Gamma(a_i)}\, H^{1\ p}_{p\ q+1}[z:\{(1-a_i,1)\};$$

$$(0,1),\ \{(1-b_j,1)\}]$$

for $p \leq q$ or for $p=q+1$ and $|z|<1$.

(Generalized Hypergeometric functions)

$$ {}_p\Psi_q\left[\begin{matrix}\{(a_i,A_i)\}\\ \{(b_j,B_j)\}\end{matrix};\ -z\right]$$

$$= H^{1\ p}_{p\ q+1}[z:\{(1-a_i,A_i)\};\ (0,1),\ \{(1-b_j,B_j)\}]$$

(Maitland's or Wright's Generalized Hypergeometric functions)

MacRobert's E-function:

$$E(p;\{a_i\};q;\{b_j\};z)$$

$$= H^{p\ 1}_{q+1\ p}[z:(1,1),\ \{(b_j,B_j)\};\ \{(a_i,A_i)\}]$$

Meijer's G-function:

$$ G^{m\ n}_{p\ q}\left[z:\begin{matrix}\{a_i\}\\ \{b_j\}\end{matrix}\right]$$

$$= H^{m\ n}_{p\ q}[z:\{(a_i,1)\};\ \{(b_j,1)\}]$$

Statistical Distributions

Bessel p.d.f.

$$f(x|\theta,\phi) = \frac{2}{\pi\theta\phi} K_o\left(\frac{x}{\theta\phi}\right)$$

$$= \frac{1}{2\pi\theta\phi} H^{2\ 0}_{0\ 2}\left[\frac{x}{2\theta\phi}:;(0,\tfrac{1}{2}),\ (0,\tfrac{1}{2})\right] \qquad x>0$$

General Hypergeometric p.d.f.

$$f(x|a,b,c,d) = \frac{da^{\frac{c}{d}}\,\Gamma(b)\,\Gamma(r-\frac{c}{d})}{\Gamma(\frac{c}{d})\,\Gamma(b-\frac{c}{d})\,\Gamma(r)}\, x^{c-1}M(b,r,-ax^d)$$

$$= \frac{a^{\frac{1}{d}}\,\Gamma(r-\frac{c}{d})}{\Gamma(\frac{c}{d})\Gamma(b-\frac{c}{d})} H^{1\ 1}_{1\ 2}\left[a^{\frac{1}{d}}x:(1-b+\tfrac{c-1}{d},\tfrac{1}{d});\right.$$

$$\left.(\tfrac{c-1}{d},\tfrac{1}{d}),\ (1-r+\tfrac{c-1}{d},\tfrac{1}{d})\right] \qquad x>0$$

APPENDIX C

Computer Program

```
      PROGRAM THESIS

**********************************************************************
**                                                                  **
**        WRITTEN BY: 1LT RALPH A. BOEDIGHEIMER                     **
**                    1LT CARL D. BODENSCHATZ                       **
**                      MS THESIS   GOR-83D                         **
**                                                                  **
**          AIR FORCE INSTITUTE OF TECHNOLOGY                       **
**                SCHOOL OF ENGINEERING                             **
**        WRIGHT-PATTERSON AIR FORCE BASE, OHIO                     **
**                                                                  **
**********************************************************************

**********************************************************************
**                                                                  **
**        PURPOSE: THIS PROGRAM ESTIMATES THE PARAMETERS            **
**                 OF THE H-FUNCTION THAT GIVES THE BEST            **
**                 FIT TO A SET OF DATA.                            **
**                                                                  **
**        INPUT/OUTPUT: TAPE8/TAPE2                                 **
**                                                                  **
**        NOTE: THIS PROGRAM IS WRITTEN IN FORTRAN 77.              **
**              IMSL MUST BE ATTACHED PRIOR TO RUNNING              **
**              THE PROGRAM FOR CALLS TO ZSPOW & GAMMA.             **
**                                                                  **
**********************************************************************

**********************************************************************
**                                                                  **
**        FLAG = 0 IF THE ZEROTH MOMENT IS USED                     **
**             = 1 IF THE ZEROTH MOMENT IS NOT USED                 **
**        M = THE NUMBER OF "B" TERMS IN THE NUMERATOR              **
**        N = THE NUMBER OF "A" TERMS IN THE NUMERATOR              **
**        P = THE TOTAL NUMBER OF "A" TERMS                         **
**        Q = THE TOTAL NUMBER OF "B" TERMS                         **
**        TYPE = 0 IF MOMENTS ARE INPUT                             **
**             = 1 IF UNIVARIATE DATA ARE INPUT                     **
**             = 2 IF ORDERED PAIRS FROM A RELATIVE                 **
**                 FREQUENCY ARE INPUT                              **
**             = 3 IF ORDERED PAIRS FROM A FUNCTION                 **
**                 ARE INPUT                                        **
**        GUESS = 0 IF THE USER WISHES TO SUPPLY HIS                **
**                  OWN INITIAL PARAMETER ESTIMATES                 **
**              = 1 IF DEFAULT INITIAL GUESSES ARE DESIRED          **
**        NUM = THE NUMBER OF MOMENTS IF TYPE = 0                   **
**            = THE NUMBER OF DATA POINTS IF TYPE = 1               **
**            = THE NUMBER OF DATA PAIRS IF TYPE = 2 OR 3           **
**                                                                  **
**********************************************************************
```

```
**********************************************************
**                                                      **
**        THE DATA MUST BE INPUT IN TAPE8 IN THE FORM:  **
**           1) FLAG,M,N,P,Q,GUESS,NUM (ALL INTEGER)    **
**           2) DATA (REAL) -                           **
**               A) TYPE 0 - 2(P+Q)+2 MOMENTS FROM      **
**                           LOWEST TO HIGHEST          **
**               B) TYPE 1 - NUM UNIVARIATE DATA        **
**               C) TYPE 2 OR 3 - NUM DATA PAIRS IN     **
**                                THE FORM X,F(X)       **
**           3) INITIAL GUESSES (REAL) -                **
**               A) GUESS OF 0 - 2(P+Q) PARAMETERS:     **
**                               'B' PAIRS IN NUMERATOR **
**                               'A' PAIRS IN NUMERATOR **
**                               'B' PAIRS IN DENOMINATOR**
**                               'A' PAIRS IN DENOMINATOR**
**               B) GUESS OF 1 - NO INPUT NECESSARY     **
**                                                      **
**********************************************************

**********************************************************
**                                                      **
**        EQS = THE NUMBER OF EQUATIONS & UNKNOWNS      **
**        IER = THE NUMBER OF ANY ERROR MESSAGE         **
**        ITMAX = THE MAXIMUM NUMBER OF ITERATIONS      **
**        NSIG = THE NUMBER OF SIGNIFICANT DIGITS       **
**        FNORM = THE NORM OF THE F EQUATION VECTOR     **
**        PAR = A VECTOR CONTAINING FLAG,M,N,P,Q,       **
**              AND THE 2(P+Q)+2 MOMENTS                **
**        X = A VECTOR OF VARIABLES BEING ESTIMATED     **
**            (I.E. THE 'A' AND 'B' PAIRS)              **
**                                                      **
**********************************************************

      INTEGER CC,EQS,IER,ITMAX,NSIG,FLAG,M,N,P,Q,
     +        TYPE,GUESS,NUM,I,J,K,L,T,U,V,W
      REAL FNORM,PAR(0:16),WK(418),X(10),PAIR(4000,2),
     +     DEV(4000),DELTAX,TEST,START1,START2,ANSWER

**********************************************************
**                                                      **
**        WK IS A WORK VECTOR WHOSE SIZE IS             **
**        DEFINED BY THE FOLLOWING FORMULA:             **
**                SIZE = (EQS/2)*((3*EQS)+15)           **
**                                                      **
**********************************************************

      EXTERNAL FCN,COMPIS
```

```
**********************************************************
**                                                      **
**       ALL VARIABLES ARE INITIALIZED.                 **
**                                                      **
**********************************************************

      READ (8,*) PAR(0),PAR(1),PAR(2),PAR(3),PAR(4),
     +           TYPE,GUESS,NUM

      FLAG=PAR(0)
      M=PAR(1)
      N=PAR(2)
      P=PAR(3)
      Q=PAR(4)
      EQS=2*(P+Q)
      NSIG=5
      ITMAX=200

**********************************************************
**                                                      **
**       THE INPUT PARAMETERS OF FLAG,M,N,P, AND Q      **
**       ARE CHECKED AGAINST SOME CONDITIONS.           **
**                                                      **
**********************************************************

      IF ((FLAG.NE.0).AND.(FLAG.NE.1)) THEN
        PRINT *,' FLAG MUST BE EITHER 0 OR 1.'
        GO TO 999
      ELSEIF ((M.GT.Q).OR.(M.LT.0)) THEN
        PRINT *,' M MUST BE BETWEEN 0 AND Q, INCLUSIVE.'
        GO TO 999
      ELSEIF ((N.GT.P).OR.(N.LT.0)) THEN
        PRINT *,' N MUST BE BETWEEN 0 AND P, INCLUSIVE.'
        GO TO 999
      ELSEIF (((P+Q).GT.5).OR.((P+Q).LT.1)) THEN
        PRINT *,' (P+Q) MUST BE BETWEEN 1 AND 5.'
        GO TO 999
      ENDIF

**********************************************************
**                                                      **
**         THE MOMENTS ARE READ IN FROM TAPE 8.  IF     **
**         ONLY RAW DATA ARE AVAILABLE, THEN THE        **
**         MOMENTS ARE CALCULATED.                      **
**                                                      **
**********************************************************
```

```
      IF (TYPE.EQ.0) THEN
        IF (NUM.NE.(EQS+2)) THEN
          PRINT *,' THE NUMBER OF MOMENTS IS INCORRECT.'
          GO TO 999
        ENDIF
        READ (8,*)(PAR(I+4),I=1,EQS+2)
      ELSEIF (TYPE.EQ.1) THEN
        IF (NUM.LT.20) THEN
          PRINT *,' FEWER THAN 20 DATA POINTS WILL'
          PRINT *,' NOT PRODUCE ACCURATE MOMENTS.'
          GO TO 999
        ENDIF
        READ (8,*)(DEV(J),J=1,NUM)
        CALL MOM1(EQS,FLAG,NUM,DEV,PAR)
      ELSEIF ((TYPE.EQ.2).OR.(TYPE.EQ.3)) THEN
        IF (NUM.LT.20) THEN
          PRINT *,' FEWER THAN 20 DATA POINTS WILL'
          PRINT *,' NOT PRODUCE ACCURATE MOMENTS.'
          GO TO 999
        ENDIF
        READ (8,*)((PAIR(K,L),L=1,2),K=1,NUM)
        DELTAX=PAIR(2,1)-PAIR(1,1)
        DO 5 J=3,NUM,1
          TEST=PAIR(J,1)- PAIR(J-1,1)
          IF (ABS(TEST-DELTAX).GT.(.5E-5)) THEN
            PRINT *,' DELTA X MUST BE THE SAME.'
            GO TO 999
          ENDIF
5       CONTINUE
        IF (TYPE.EQ.2) THEN
          CALL MOM3(EQS,FLAG,NUM,PAIR,1.0,PAR)
        ELSEIF (TYPE.EQ.3) THEN
          CALL MOM3(EQS,FLAG,NUM,PAIR,DELTAX,PAR)
          IF (FLAG.EQ.0) THEN
            PRINT *,' THE ZEROTH MOMENT IS ',PAR(5)
            PRINT *,' IF YOU KNOW THE DATA IS FROM A'
            PRINT *,' STATISTICAL DISTRIBUTION AND WANT'
            PRINT *,' THE ZEROTH MOMENT TO EQUAL ONE,'
            PRINT *,' THEN TYPE A ONE.'
            PRINT *,' OTHERWISE, TYPE ANY OTHER NUMBER.'
            READ *,ANSWER
            IF (ANSWER.EQ.1.0) THEN
              PAR(5)=1.0
            ENDIF
          ENDIF
        ENDIF
      ELSE
        PRINT *,' TYPE MUST BE EITHER 0,1,2, OR 3.'
        GO TO 999
      ENDIF
```

```
**********************************************************
**                                                      **
**       THE INITIAL GUESS TO THE VECTOR X IS READ IN.  **
**       DEFAULT VALUES ARE AVAILABLE WHICH ENSURE THE  **
**       CONVERGENCE CONDITIONS ARE SATISFIED INITIALLY  **
**       BY MAKING D IN SUBROUTINE CHKR GREATER THAN 0.  **
**                                                      **
**********************************************************

      IF (GUESS.EQ.0) THEN
        READ (8,*)(X(I),I=1,EQS)
      ELSEIF (GUESS.EQ.1) THEN
        START1=.7654321
        START2=.87654321
        DO 10 T=1,2*M-1,2
          X(T)=START1
          X(T+1)=START2
          START1=START1-.1
          START2=START2-.1
10      CONTINUE
        DO 20 U=T,2*(M+N)-1,2
          X(U)=START1
          X(U+1)=START2
          START1=START1-.1
          START2=START2-.1
20      CONTINUE
        DO 30 V=U,2*(Q+N)-1,2
          X(V)=START1
          X(V+1)=START2
          START1=START1-.1
          START2=START2-.1
30      CONTINUE
        DO 40 W=V,2*(P+Q)-1,2
          X(W)=START1
          X(W+1)=START2
          START1=START1-.1
          START2=START2-.1
40      CONTINUE
      ELSE
        PRINT *,' VARIABLE GUESS MUST BE EITHER 0 OR 1.'
        GO TO 999
      ENDIF
```

```
**********************************************************
**                                                      **
**       THE INITIAL GUESS OF THE VECTOR X.IS           **
**       CHECKED AGAINST THE CONVERGENCE CONDITIONS.    **
**                                                      **
**********************************************************

      CALL CHKR(EQS,M,N,P,Q,X,CC)
      IF (CC.EQ.0) THEN
        PRINT *,' A NEW INITIAL GUESS IS NEEDED.'
        GO TO 999
      ENDIF
      PRINT *,' THE INITIAL GUESS MEETS THE'
      PRINT *,' CONVERGENCE CONDITIONS.'

**********************************************************
**                                                      **
**      ZSPOW IS AN IMSL ROUTINE WHICH USES             **
**      POWELL'S METHOD TO APPROXIMATE THE              **
**      ROOTS OF A SYSTEM OF EQUATIONS.                 **
**                                                      **
**********************************************************

      CALL ZSPOW(FCN,NSIG,EQS,ITMAX,PAR,X,FNORM,WK,IER)

**********************************************************
**                                                      **
**      THE FINAL ESTIMATE OF THE VECTOR X IS           **
**      CHECKED AGAINST THE CONVERGENCE CONDITIONS.     **
**                                                      **
**********************************************************

      CALL CHKR(EQS,M,N,P,Q,X,CC)
      IF (CC.EQ.0) THEN
        PRINT *,' THE FINAL ESTIMATE OF THE X VECTOR DOES'
        PRINT *,' NOT MEET THE CONVERGENCE CONDITIONS.'
        GO TO 998
      ENDIF
      PRINT *,' THE FINAL ESTIMATE OF THE X VECTOR MEETS'
      PRINT *,' THE CONVERGENCE CONDITIONS.'

**********************************************************
**                                                      **
**      IF THE INITIAL GUESS OF THE X VECTOR            **
**      MEETS THE CONVERGENCE CONDITIONS, THEN          **
**      THE RESULTS OF ZSPOW ARE OUTPUT TO TAPE 2.      **
**                                                      **
**********************************************************

 998  CALL RESULT(EQS,FLAG,M,N,P,Q,X,PAR,FNORM)
      GO TO 1000
```

```
**************************************************
**                                              **
**      ALL ERRORS, EXCEPT A FINAL ESTIMATE THAT **
**      DOES NOT MEET CONVERGENCE CONDITIONS, ARE **
**      ROUTED TO THIS LINE WHICH ENDS THE        **
**      EXECUTION OF THE PROGRAM.                 **
**                                              **
**************************************************

 999  PRINT *,' THE PROGRAM WILL BE STOPPED BECAUSE'
      PRINT *,' OF THIS ERROR.'

1000  CONTINUE
      END

**************************************************************************
**************************************************************************
**************************************************************************
```

```
      SUBROUTINE FCN(X,F,EQS,PAR)

*****************************************************
**                                                 **
**      THIS SUBROUTINE IS NEEDED TO DEFINE THE    **
**      SYSTEM OF NONLINEAR EQUATIONS.             **
**                                                 **
*****************************************************

      INTEGER EQS,I,FLAG,M,N,P,Q
      REAL IOFS(13),PAR(0:EQS+6),F(EQS),X(EQS)

      FLAG=PAR(0)
      M=PAR(1)
      N=PAR(2)
      P=PAR(3)
      Q=PAR(4)

      CALL COMPIS(EQS,FLAG,M,N,P,Q,X,IOFS)

      DO 10 I=1,EQS,1
        F(I)=(PAR(I+4)*PAR(I+6)*IOFS(I+1+FLAG)**2)/
     +    (PAR(I+5)**2*IOFS(I+FLAG)*IOFS(I+2+FLAG)) - 1.0
 10   CONTINUE

      RETURN
      END
```

```
      SUBROUTINE RESULT(EQS,FLAG,M,N,P,Q,X,PAR,FNORM)

*****************************************************
**                                                 **
**      THIS SUBROUTINE PRINTS THE SOLUTION OF     **
**      THE SYSTEM OF NONLINEAR EQUATIONS.         **
**      C AND K ARE VARIABLES ESTIMATED FROM X.    **
**                                                 **
*****************************************************

      INTEGER EQS,FLAG,M,N,P,Q,T,U,V,W,CNTA,CNTB
      REAL IOFS(13),X(EQS),PAR(0:EQS+6),FNORM,C,K

      CNTA=0
      CNTB=0

      WRITE(2,'(" RESULTS OF ZSPOW -")')
      WRITE(2,'("  NUMERATOR:")')
      DO 10 T=1,2*M-1,2
        CNTB=CNTB+1
        WRITE(2,'("     SMALLB(",I1,")=",F25.18)')CNTB,X(T)
        WRITE(2,'("     BIGB(",I1,")=",F25.18)')CNTB,X(T+1)
10    CONTINUE
      DO 20 U=T,2*(M+N)-1,2
        CNTA=CNTA+1
        WRITE(2,'("     SMALLA(",I1,")=",F25.18)')CNTA,X(U)
        WRITE(2,'("     BIGA(",I1,")=",F25.18)')CNTA,X(U+1)
20    CONTINUE
      WRITE(2,'("  DENOMINATOR:")')
      DO 30 V=U,2*(Q+N)-1,2
        CNTB=CNTB+1
        WRITE(2,'("     SMALLB(",I1,")=",F25.18)')CNTB,X(V)
        WRITE(2,'("     BIGB(",I1,")=",F25.18)')CNTB,X(V+1)
30    CONTINUE
      DO 40 W=V,2*(P+Q)-1,2
        CNTA=CNTA+1
        WRITE(2,'("     SMALLA(",I1,")=",F25.18)')CNTA,X(W)
        WRITE(2,'("     BIGA(",I1,")=",F25.18)')CNTA,X(W+1)
40    CONTINUE

      CALL COMPIS(EQS,FLAG,M,N,P,Q,X,IOFS)

      C=(PAR(6-FLAG)*IOFS(3))/(PAR(7-FLAG)*IOFS(2))
      K=PAR(6-FLAG)*C**2/IOFS(2)

      WRITE(2,'(/,"  VALUES OF K & C ARE:")')
      WRITE(2,'("     K=",F25.18)')K
      WRITE(2,'("     C=",F25.18)')C
      WRITE(2,'("     FNORM=",F25.18,/,/)')FNORM

      RETURN
      END
```

```
      SUBROUTINE COMPIS(EQS,FLAG,M,N,P,Q,X,IOFS)

*****************************************************************
**                                                             **
**       THIS SUBROUTINE COMPUTES THE VECTOR I(S):             **
**       PRODUCTS AND QUOTIENTS OF GAMMA FUNCTIONS FOR         **
**       A GIVEN VALUE OF S.                                   **
**                                                             **
*****************************************************************

      INTEGER EQS,FLAG,M,N,P,Q,R,S,T,U,V,W
      REAL X(EQS),IOFS(13),GAMMA,BNUM,BDEN,ANUM,ADEN

      DO 5 R=1,13,1
        IOFS(R)=0.0
5     CONTINUE

      DO 50 S=1+FLAG,EQS+2+FLAG,1
        BNUM=1.0
        BDEN=1.0
        ANUM=1.0
        ADEN=1.0
        DO 10 T=1,2*M-1,2
          BNUM=BNUM*GAMMA(X(T)+S*X(T+1))
10      CONTINUE
        DO 20 U=T,2*(M+N)-1,2
          ANUM=ANUM*GAMMA(1-X(U)-S*X(U+1))
20      CONTINUE
        DO 30 V=U,2*(Q+N)-1,2
          BDEN=BDEN*GAMMA(1-X(V)-S*X(V+1))
30      CONTINUE
        DO 40 W=V,2*(P+Q)-1,2
          ADEN=ADEN*GAMMA(X(W)+S*X(W+1))
40      CONTINUE
        IOFS(S)=(BNUM*ANUM)/(BDEN*ADEN)
50    CONTINUE

      RETURN
      END
```

```
      SUBROUTINE CHKR(EQS,M,N,P,Q,X,CC)

*****************************************************
**                                                 **
**       THIS SUBROUTINE CHECKS THE CONVERGENCE    **
**       CONDITIONS FOR THE INITIAL GUESS AND THE  **
**       FINAL ESTIMATE OF THE X VECTOR.           **
**                                                 **
*****************************************************

      INTEGER CC,EQS,M,N,P,Q,T,U,V,W
      REAL SUMSBN,SUMBBN,SUMSAN,SUMBAN,SUMSBD,SUMBBD,
     +     SUMSAD,SUMBAD,X(EQS),TEST1,TEST2,WLOW,WHIGH,
     +     EWLOW,EWHIGH,D,E,L

      SUMSBN=0.0
      SUMBBN=0.0
      SUMSAN=0.0
      SUMBAN=0.0
      SUMSBD=0.0
      SUMBBD=0.0
      SUMSAD=0.0
      SUMBAD=0.0
      WLOW=-10000.0
      WHIGH=10000.0

      DO 10 T=1,2*M-1,2
        SUMSBN=SUMSBN+X(T)
        SUMBBN=SUMBBN+X(T+1)
        TEST1=-X(T)/X(T+1)
        IF ((TEST1-WLOW).GT.(.5E-5)) THEN
          WLOW=TEST1
        ENDIF
10    CONTINUE
      DO 20 U=T,2*(M+N)-1,2
        SUMSAN=SUMSAN+X(U)
        SUMBAN=SUMBAN+X(U+1)
        TEST2=(1-X(U))/X(U+1)
        IF ((TEST2-WHIGH).LT.(-.5E-5)) THEN
          WHIGH=TEST2
        ENDIF
20    CONTINUE
      DO 30 V=U,2*(Q+N)-1,2
        SUMSBD=SUMSBD+X(V)
        SUMBBD=SUMBBD+X(V+1)
30    CONTINUE
      DO 40 W=V,2*(P+Q)-1,2
        SUMSAD=SUMSAD+X(W)
        SUMBAD=SUMBAD+X(W+1)
40    CONTINUE
```

```
      D=SUMBAN+SUMBBN-SUMBAD-SUMBBD
      E=(SUMBAN+SUMBAD)-(SUMBBN+SUMBBD)
      L=(SUMSBN+SUMSBD)-(.5*Q)-(SUMSAN+SUMSAD)+(.5*P)

      EWLOW=E*WLOW
      EWHIGH=E*WHIGH

      IF ((WLOW-WHIGH).LT.(-.5E-5)) THEN
        IF (D.GT.(.5E-5)) THEN
          CC=1
        ELSEIF (D.LT.(-.5E-5)) THEN
          CC=0
          PRINT *,' CASE 5'
        ELSEIF ((ABS(D).LE.(.5E-5)).AND.(E.LT.(-.5E-5))) THEN
          IF ((L-EWLOW).LT.(-.5E-5)) THEN
            CC=1
          ELSEIF ((L-EWLOW).GT.(.5E-5)) THEN
            CC=0
            PRINT *,' CASE 1'
          ELSE
            CC=1
          ENDIF
        ELSEIF ((ABS(D).LE.(.5E-5)).AND.(E.GT.(.5E-5))) THEN
          IF ((L-EWHIGH).LT.(-.5E-5)) THEN
            CC=1
          ELSEIF ((L-EWHIGH).GT.(.5E-5)) THEN
            CC=0
            PRINT *,' CASE 2'
          ELSE
            CC=1
          ENDIF
        ELSEIF ((ABS(D).LE.(.5E-5)).AND.(ABS(E).LE.(.5E-5)))
     +  THEN
          IF (L.LT.(-.5E-5)) THEN
            CC=1
          ELSEIF (L.GT.(.5E-5)) THEN
            CC=0
            PRINT *,' CASE 3'
          ELSE
            CC=0
            PRINT *,' CASE 4'
          ENDIF
        ENDIF
      ELSE
        CC=0
        PRINT *,' CASE 6'
        PRINT *,' NO OMEGA IS POSSIBLE.'
      ENDIF

      RETURN
      END
```

```
      SUBROUTINE MOM1(EQS,FLAG,NUM,DEV,PAR)

*****************************************************
**                                                 **
**     THIS SUBROUTINE GENERATES THE MOMENTS FOR    **
**     UP TO 4000 UNIVARIATE DATA POINTS.          **
**                                                 **
*****************************************************

      INTEGER COUNT,I,J,EQS,FLAG,NUM
      REAL SUM,DEV(4000),PAR(0:EQS+6)

      COUNT=5

      DO 20 I=FLAG,EQS+1+FLAG,1
        SUM=0.0
        DO 10 J=1,NUM,1
          SUM=SUM+DEV(J)**I
 10     CONTINUE
        SUM=SUM/NUM
        PAR(COUNT)=SUM
        COUNT=COUNT+1
 20   CONTINUE

      RETURN
      END

      SUBROUTINE MOM3(EQS,FLAG,NUM,PAIR,DELTAX,PAR)

*****************************************************
**                                                 **
**     THIS SUBROUTINE GENERATES THE MOMENTS FOR    **
**     UP TO 4000 DATA PAIRS (X,F(X)).             **
**                                                 **
*****************************************************

      INTEGER COUNT,I,J,EQS,FLAG,NUM
      REAL SUM,PAIR(4000,2),DELTAX,PAR(0:EQS+6)

      COUNT=5

      DO 20 I=FLAG,EQS+1+FLAG,1
        SUM=0.0
        DO 10 J=1,NUM,1
          SUM=SUM+PAIR(J,1)**I*PAIR(J,2)
 10     CONTINUE
        SUM=SUM*DELTAX
        PAR(COUNT)=SUM
        COUNT=COUNT+1
 20   CONTINUE

      RETURN
      END
```

APPENDIX D

Comparison of Estimated H-function and Actual Data

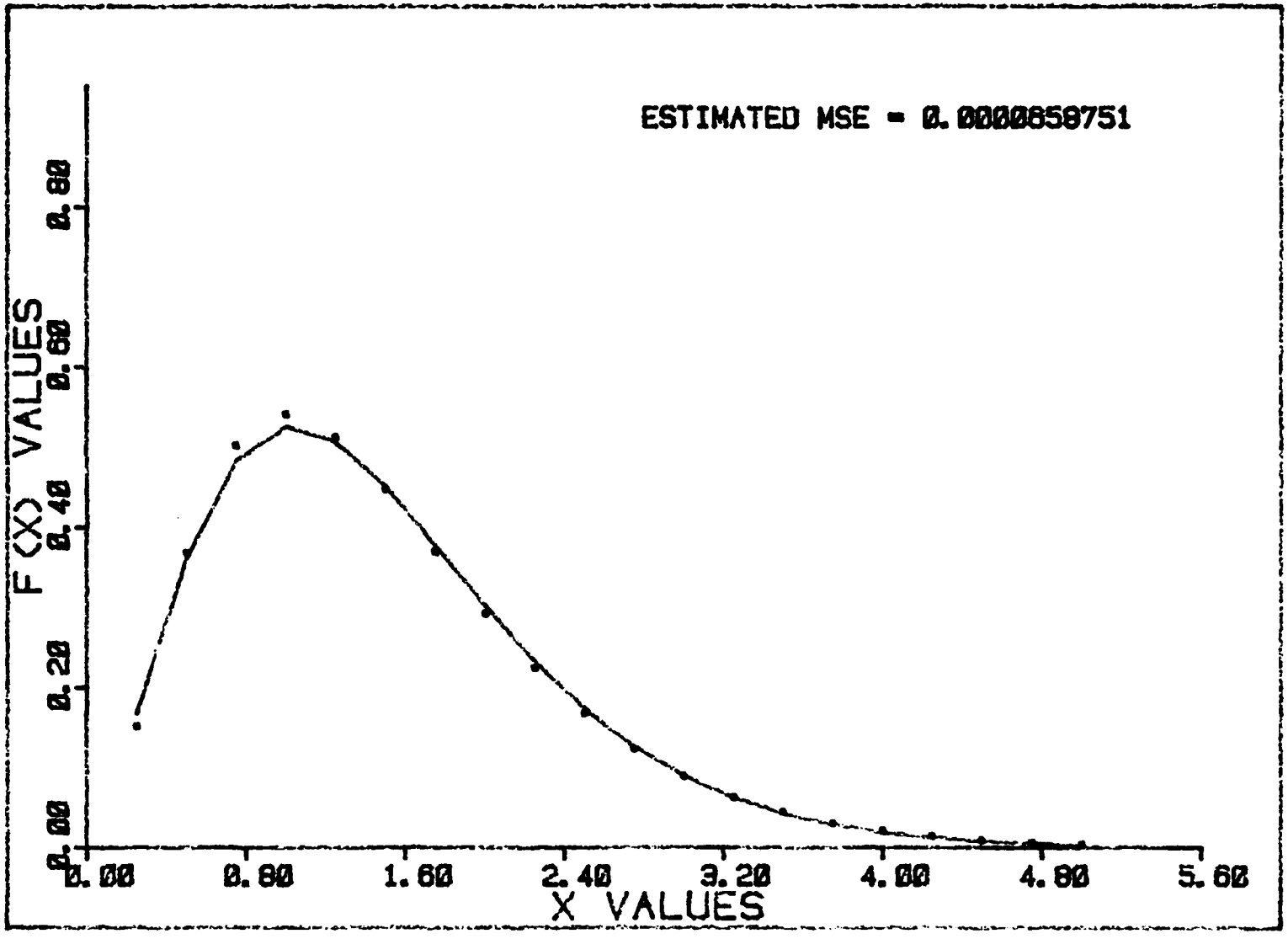

Figure 4A. Gamma (θ=3, ∅=2) n=20

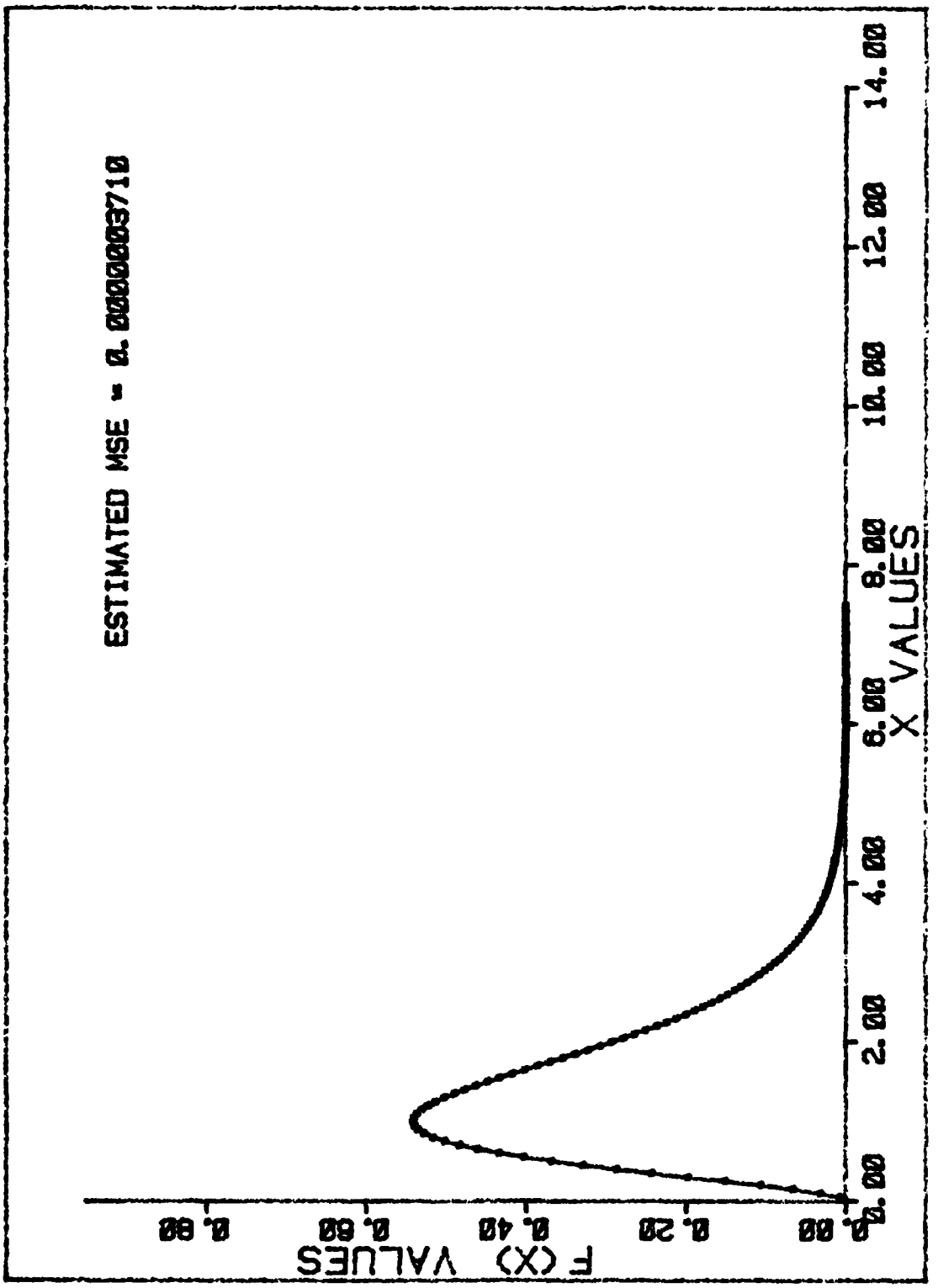

Figure 4B. Gamma (θ=3, θ=2) n=150

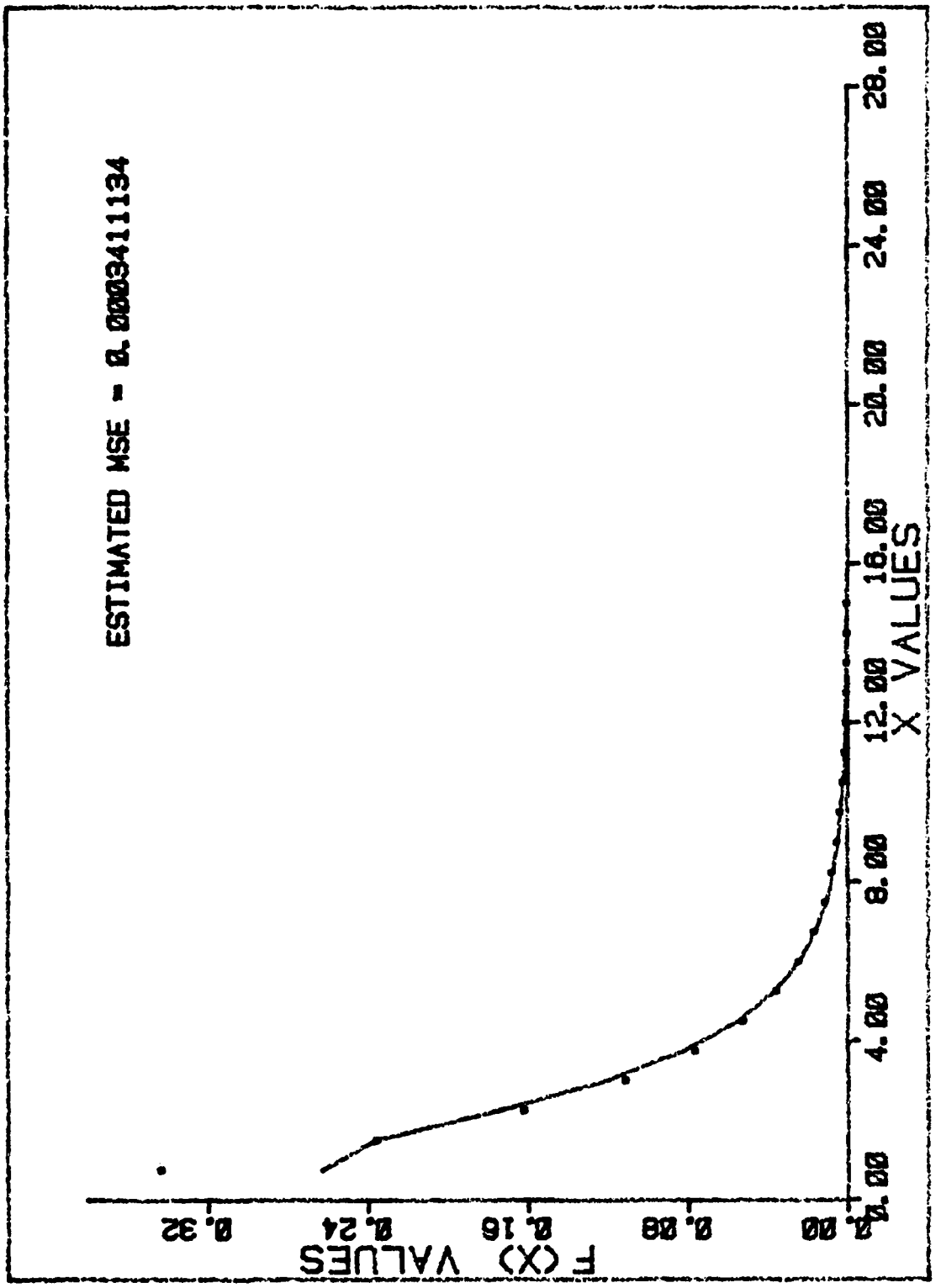

Figure 5A. Exponential (Ø=1/2) n=20

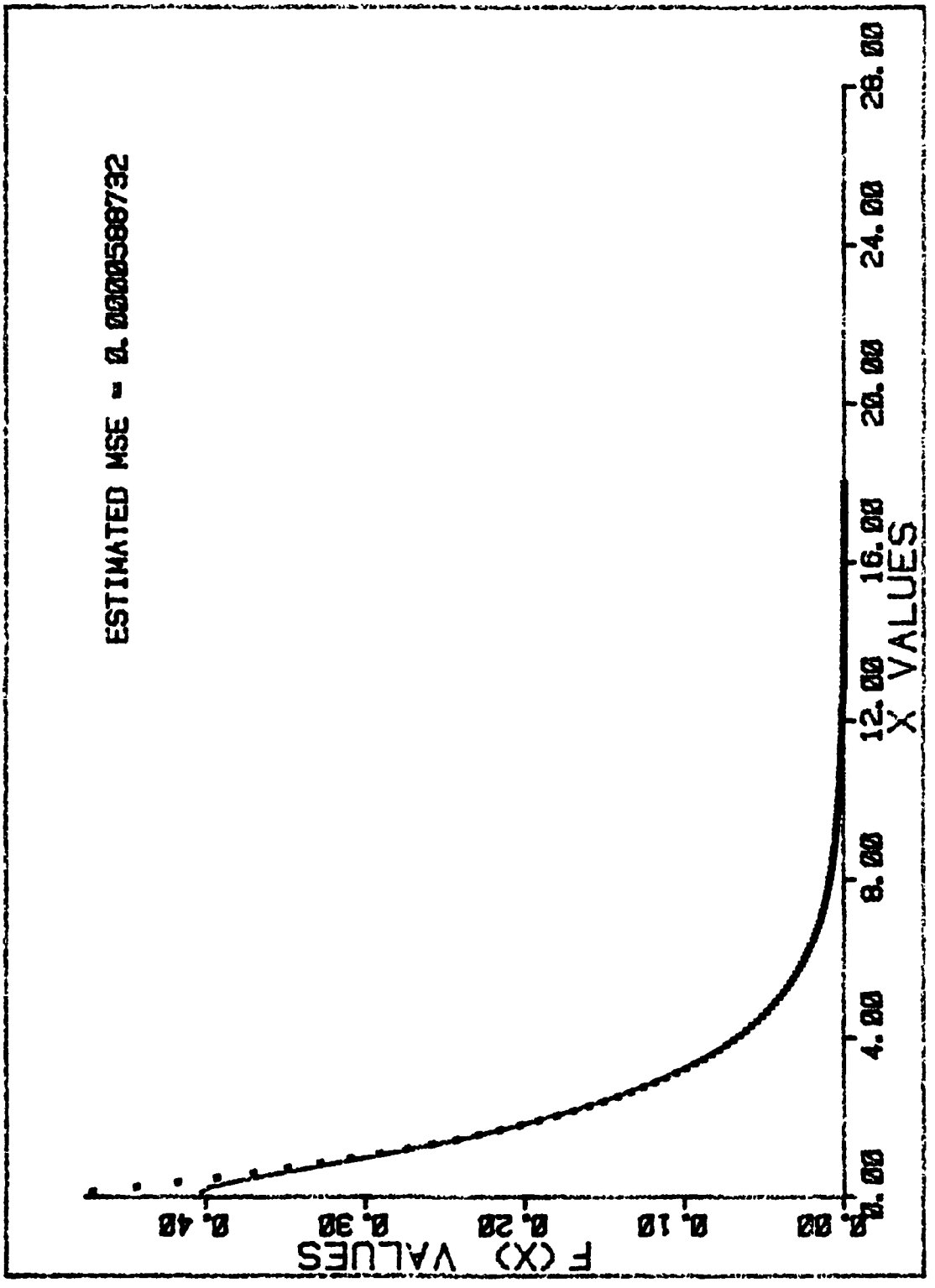

Figure 5B. Exponential (Ø=1/2) n=150

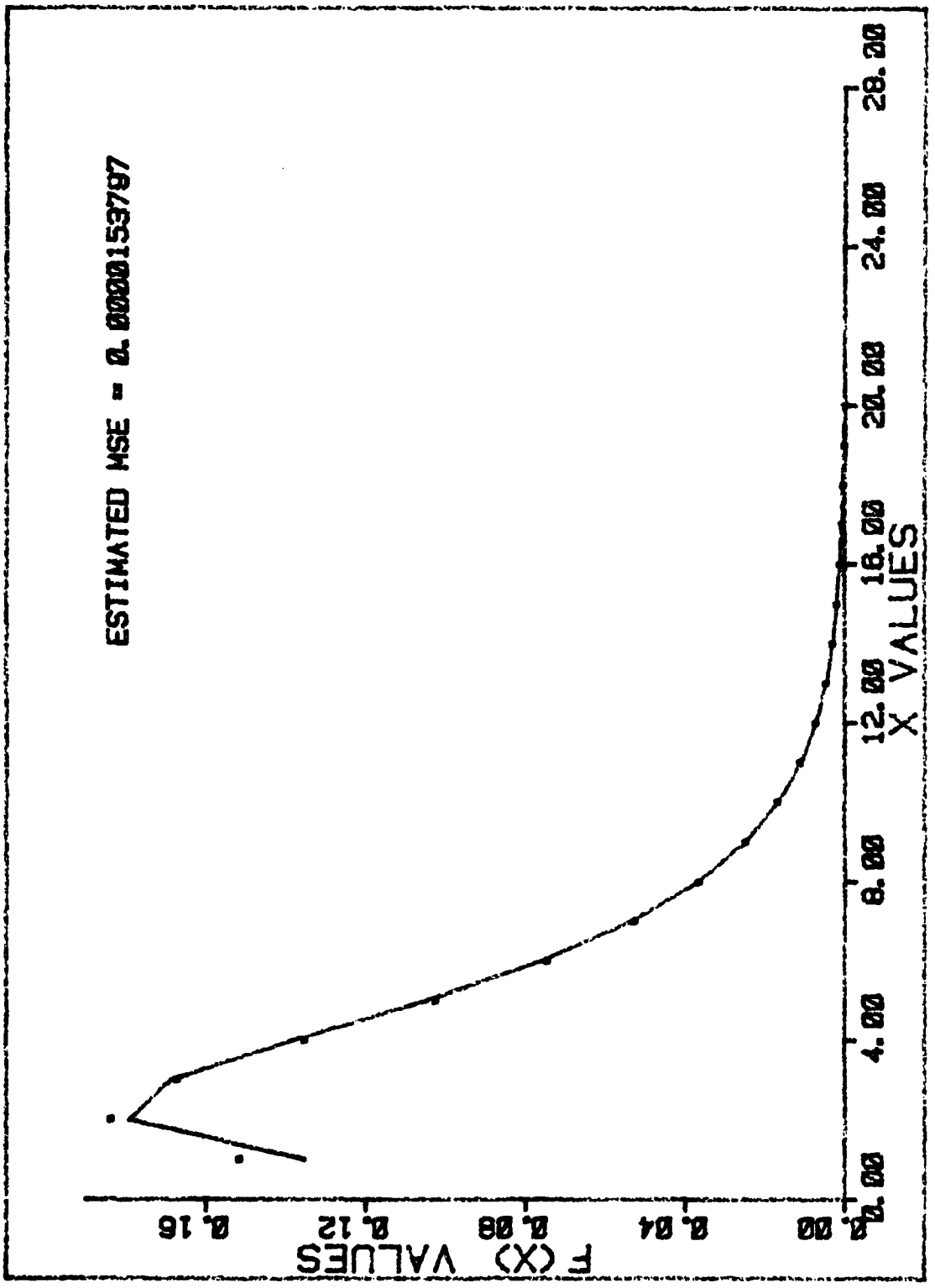

Figure 6A. Chi-Square ($\theta'=4$) n=20

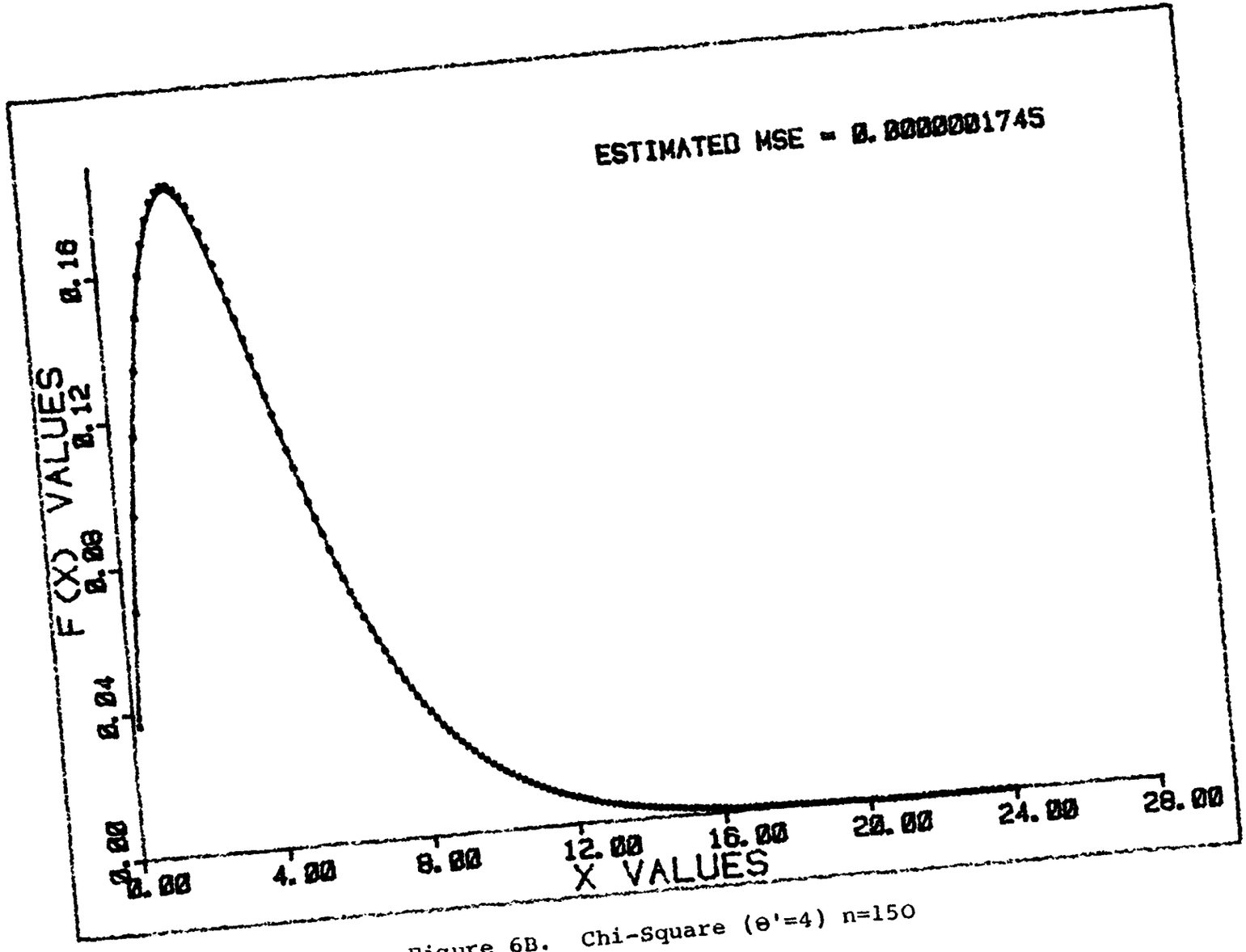

Figure 6B. Chi-Square ($\theta'=4$) n=150

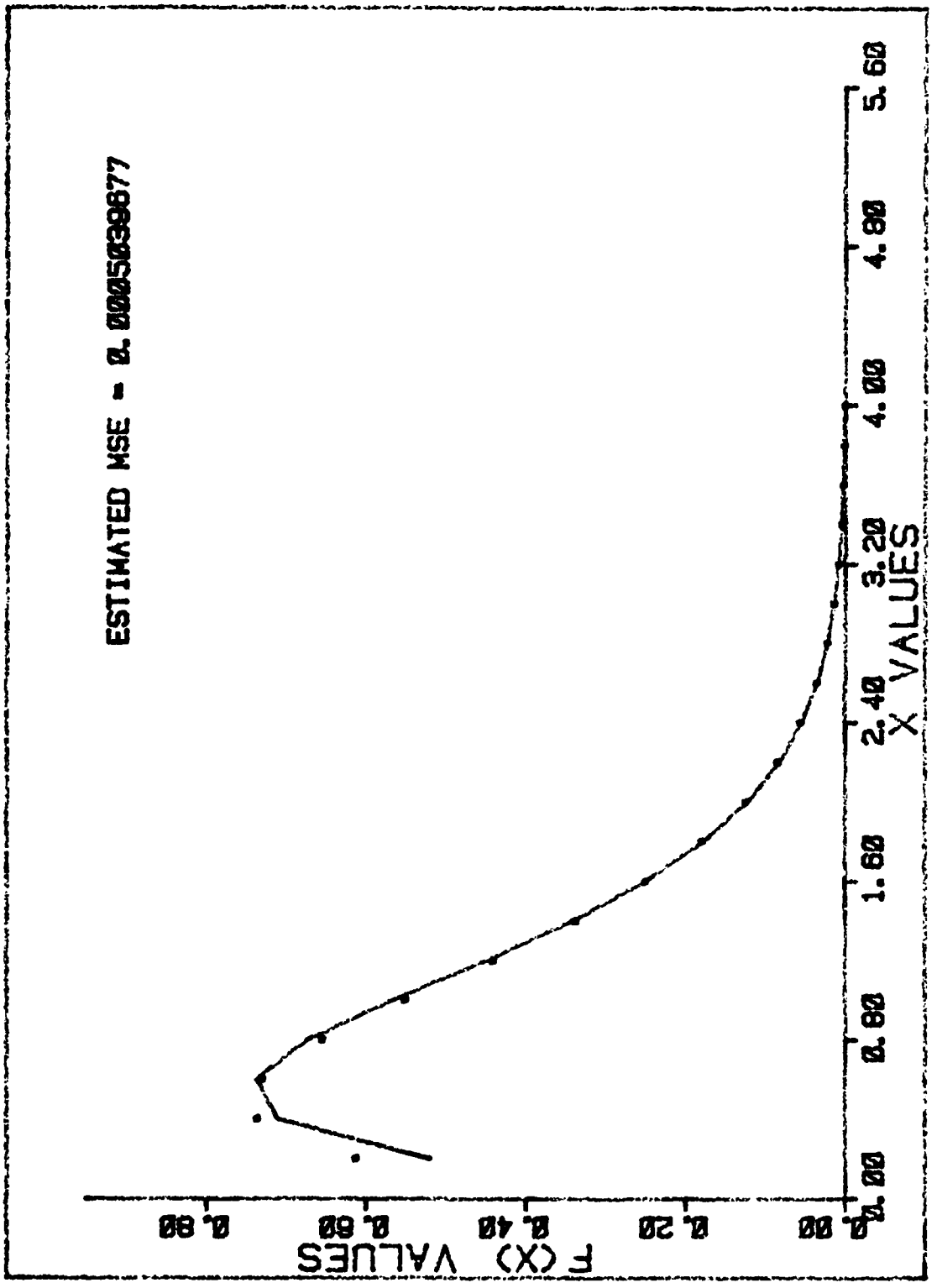

Figure 7A. Weibull (θ=3/2, Ø=1) n=20

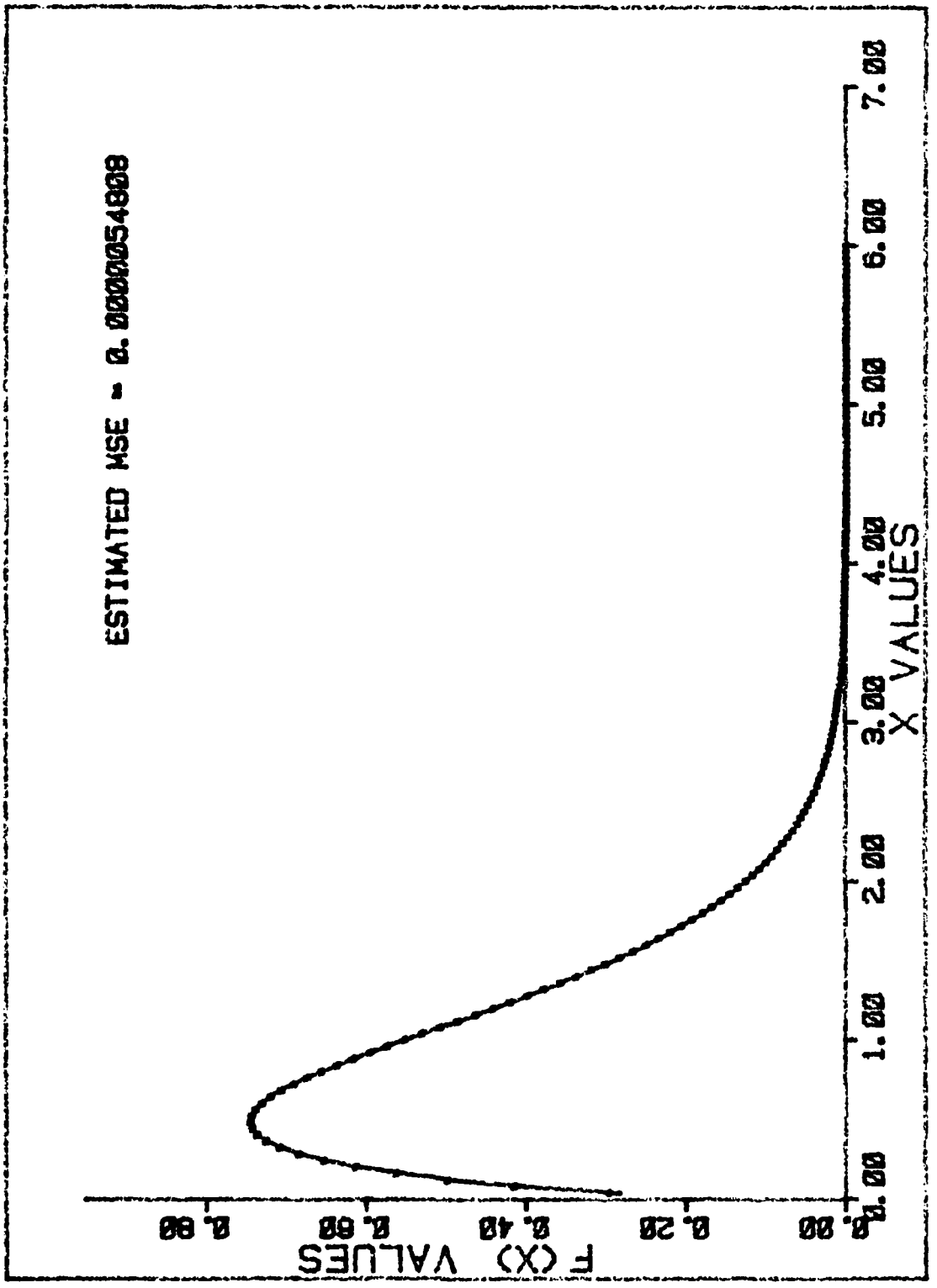

Figure 7B. Weibull (θ=3/2, Ø=1) n=150

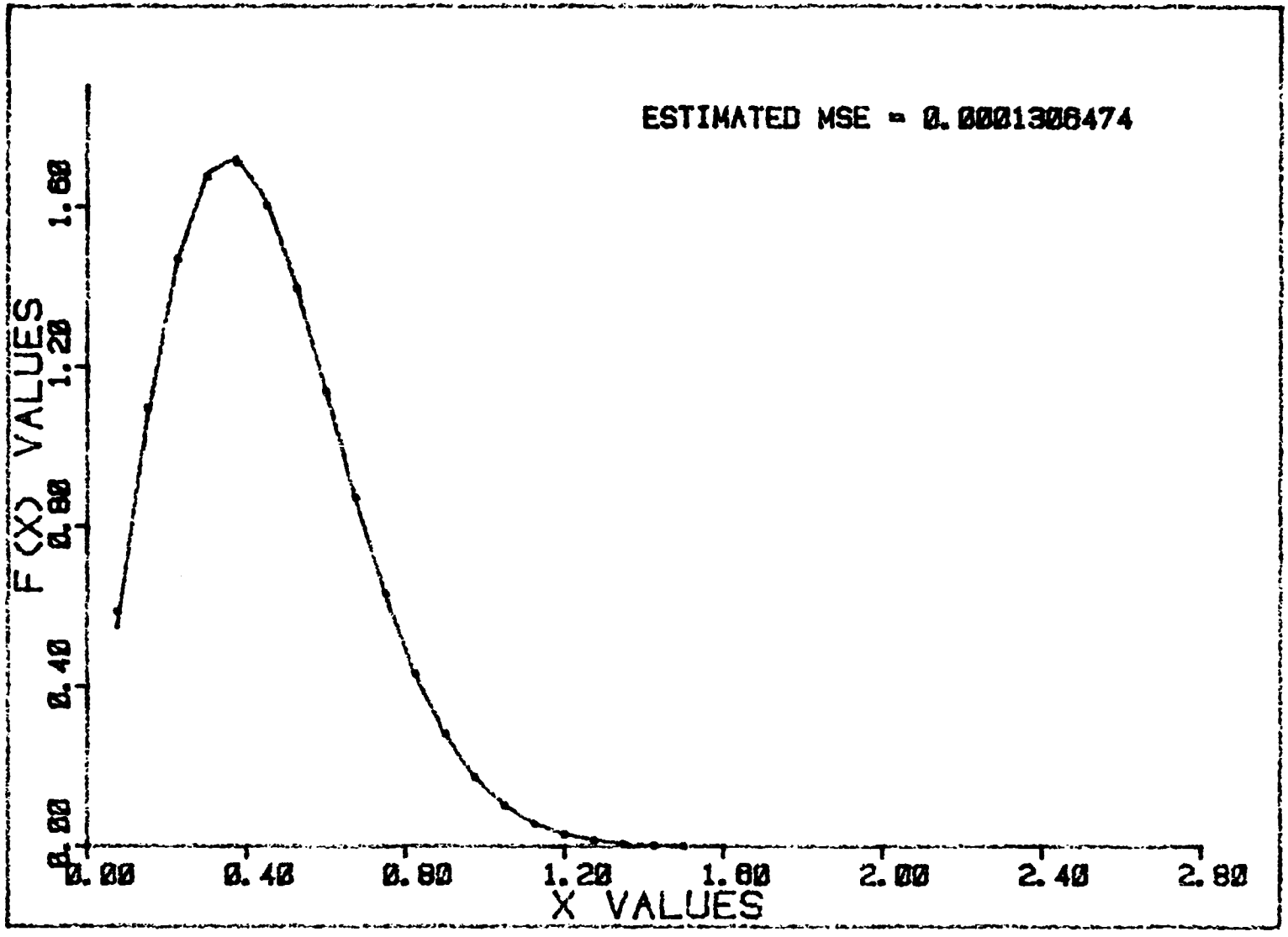

Figure 8A. Rayleigh (Ø=4) n=20

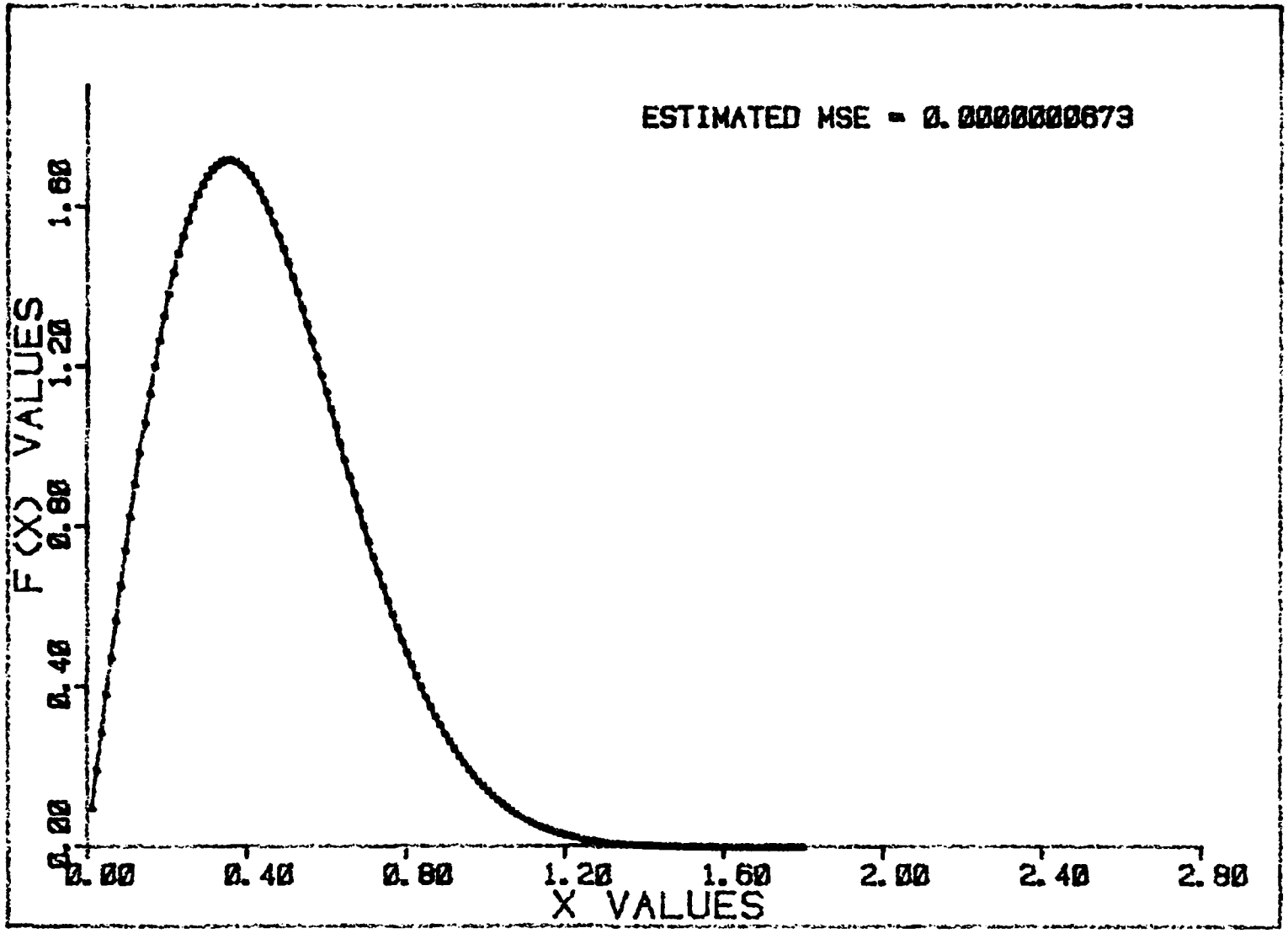

Figure 8B. Rayleigh (Ø=4) n=150

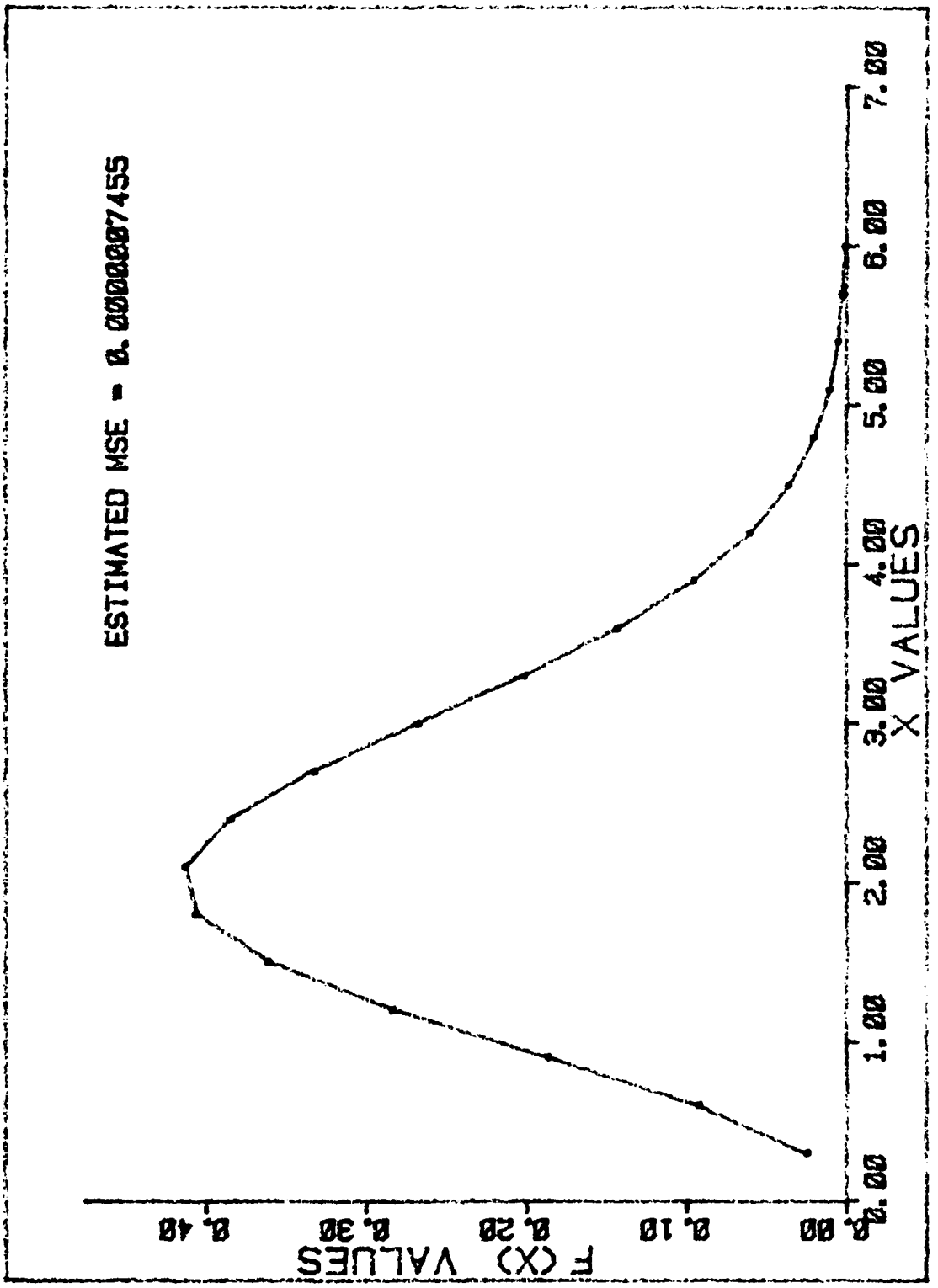

Figure 9A. Maxwell (θ=2) n=20

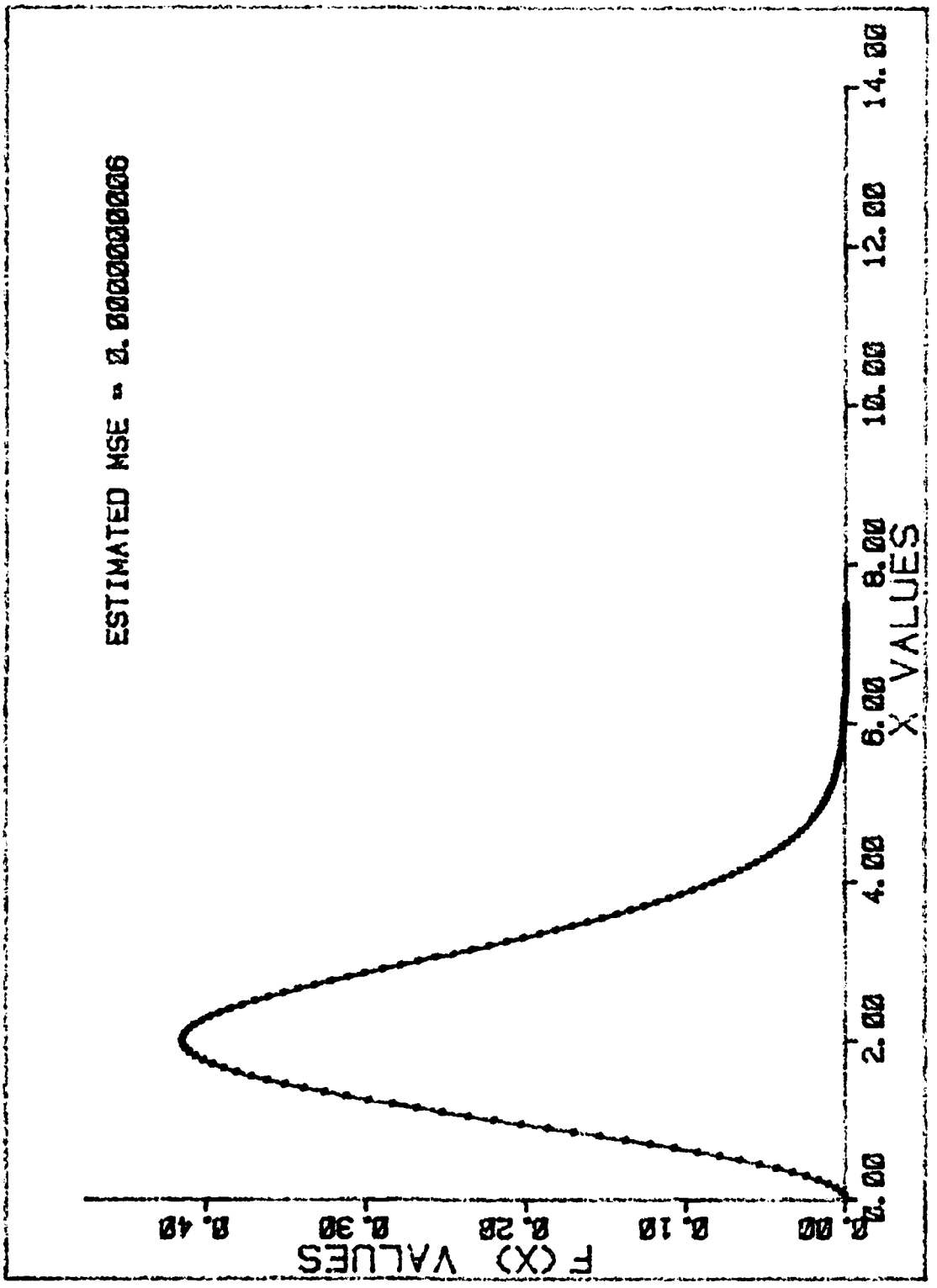

Figure 9B. Maxwell (θ=2) n=150

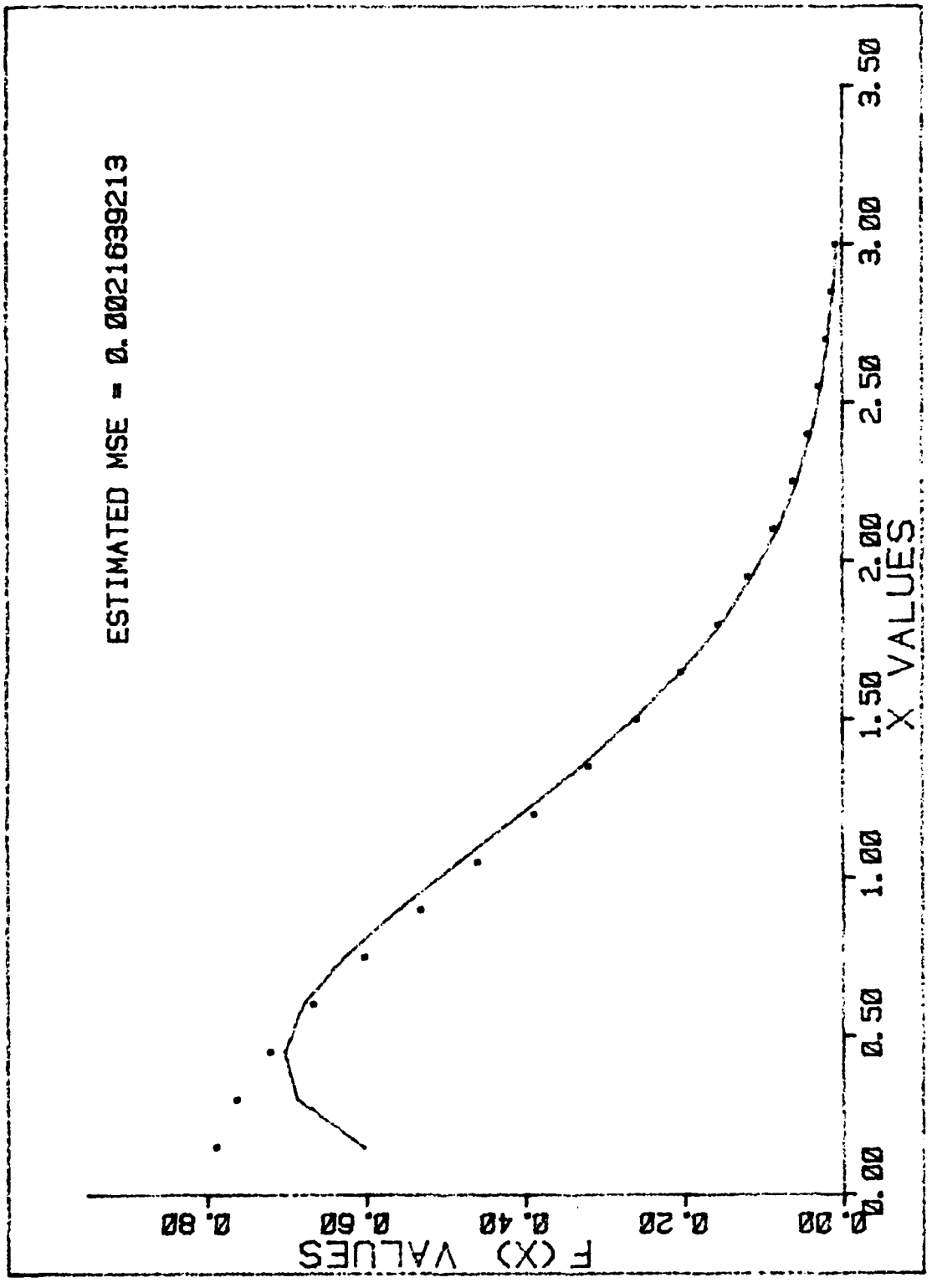

Figure 10A. Half-Normal (θ=1) n=20

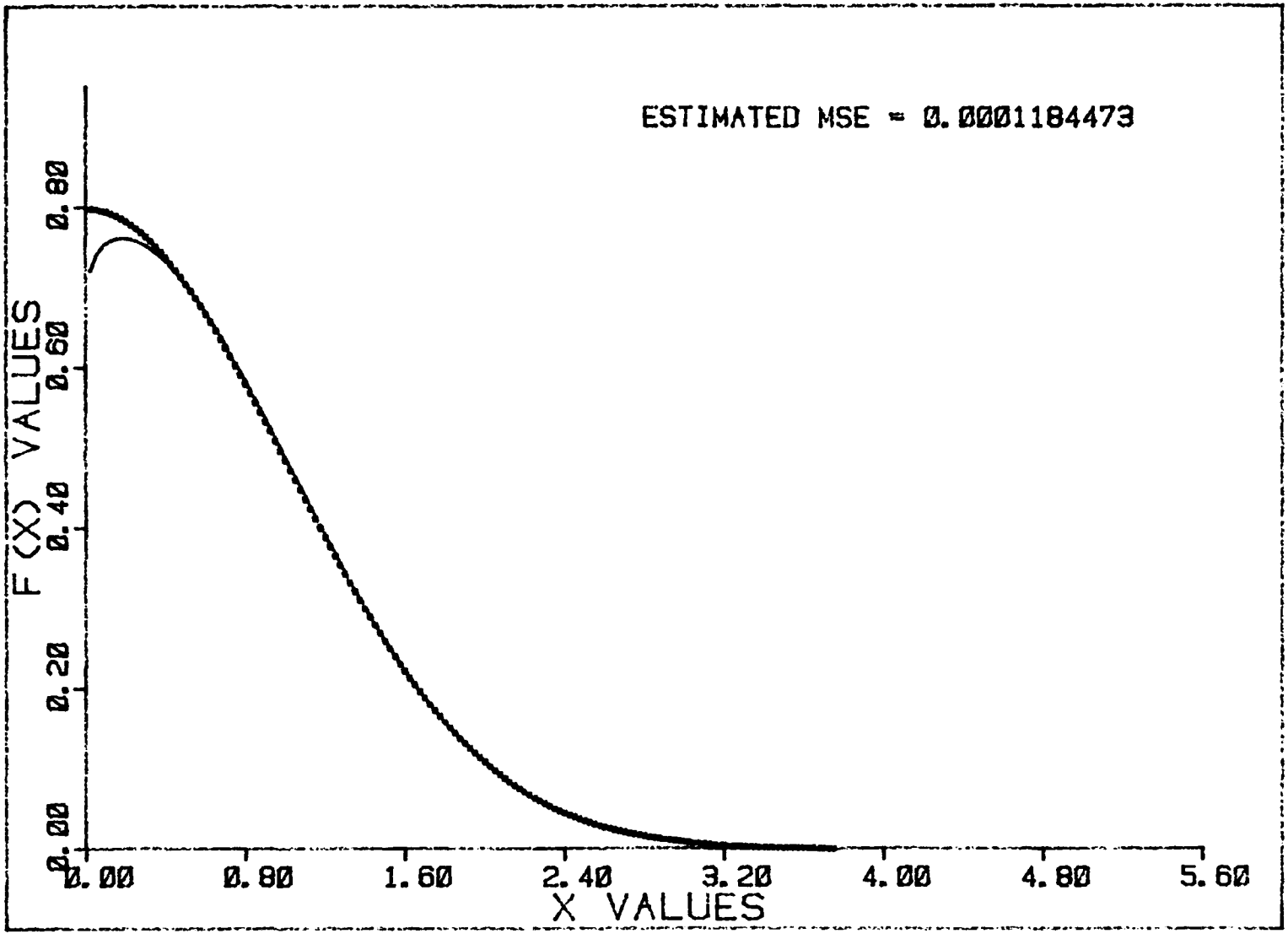

Figure 10B. Half-Normal (θ=1) n=150

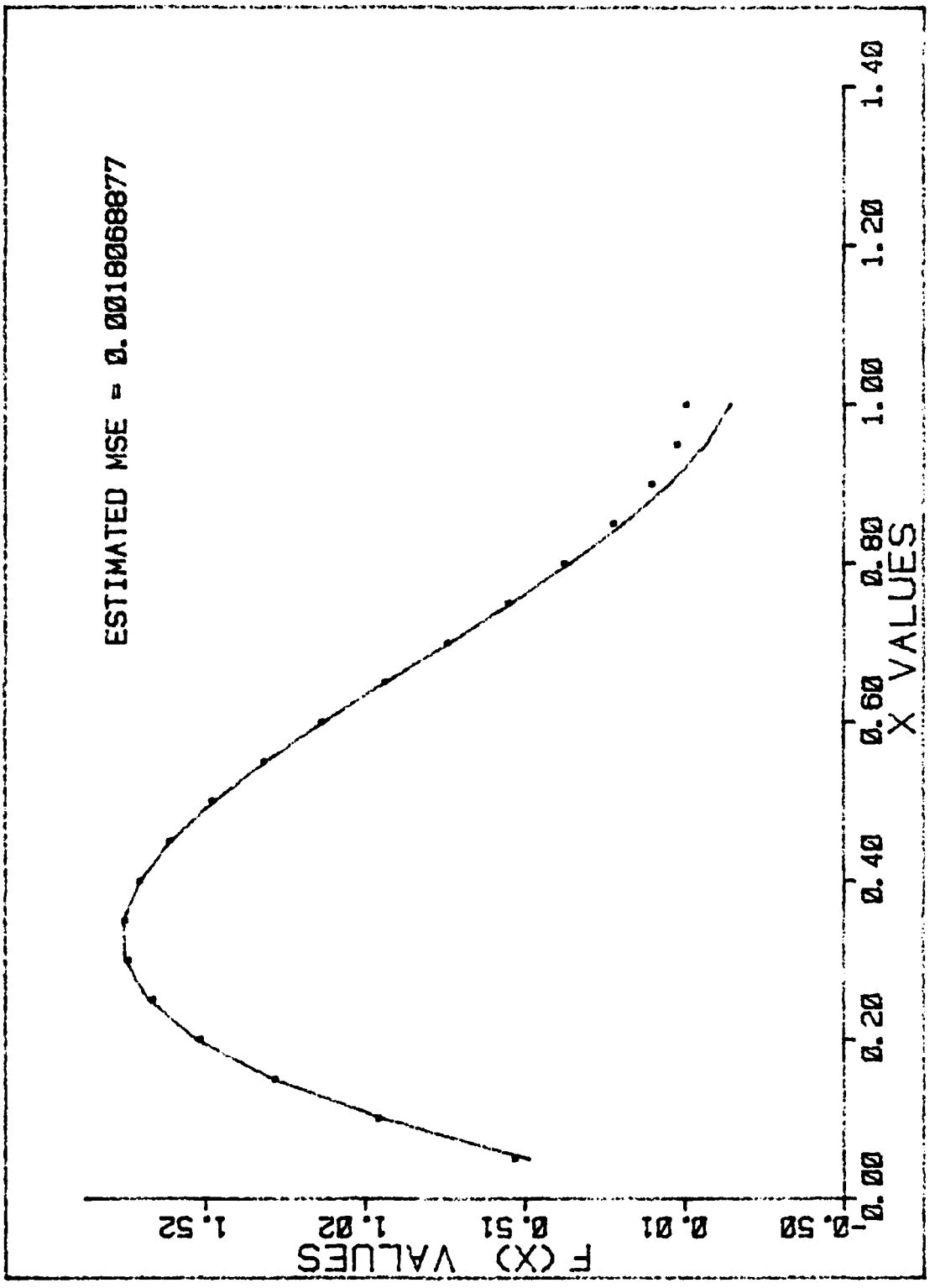

Figure 11A. Beta (θ=2, Ø=3) n=20

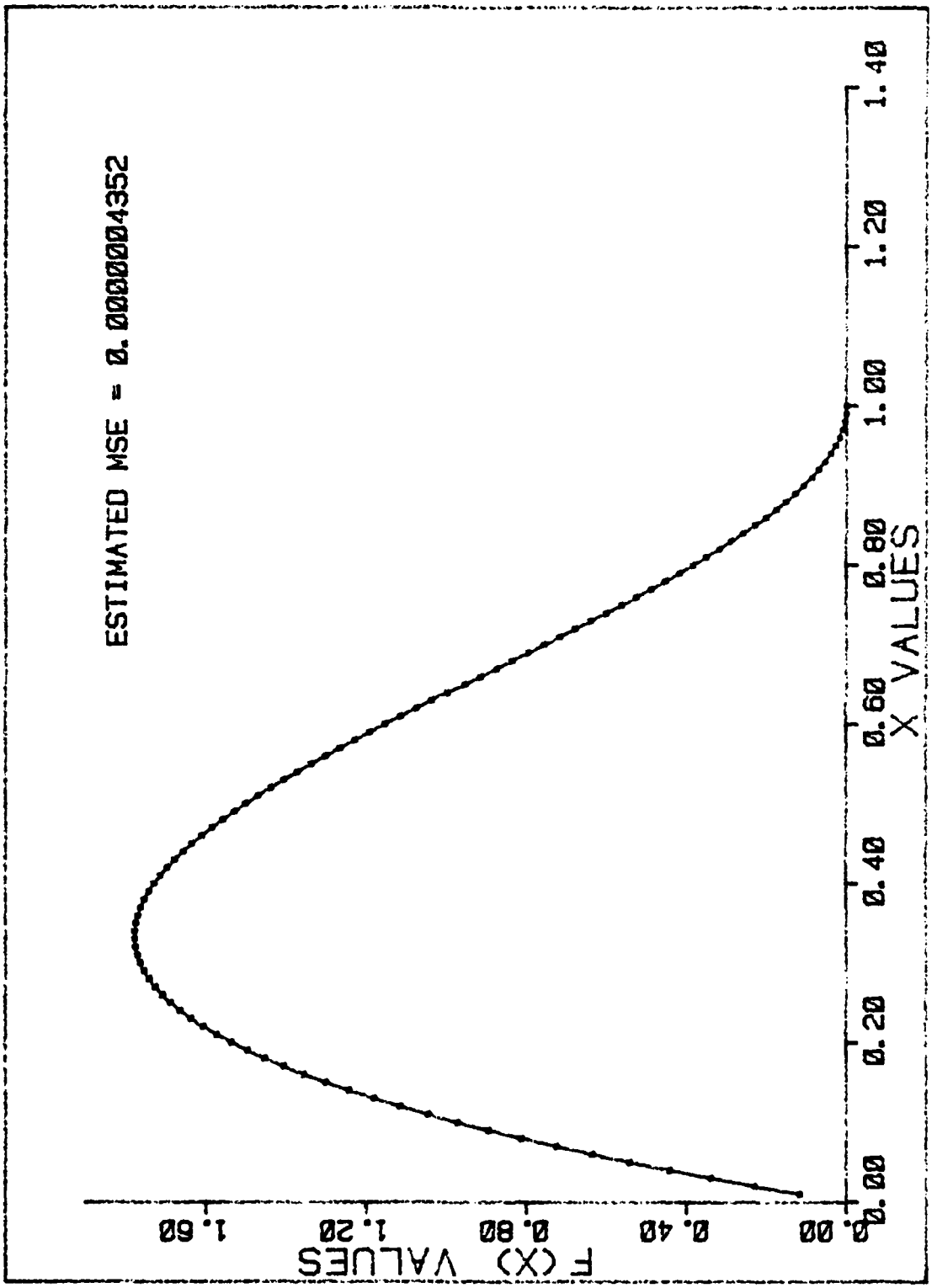

Figure 11B. Beta (θ=2, Ø=3) n=100

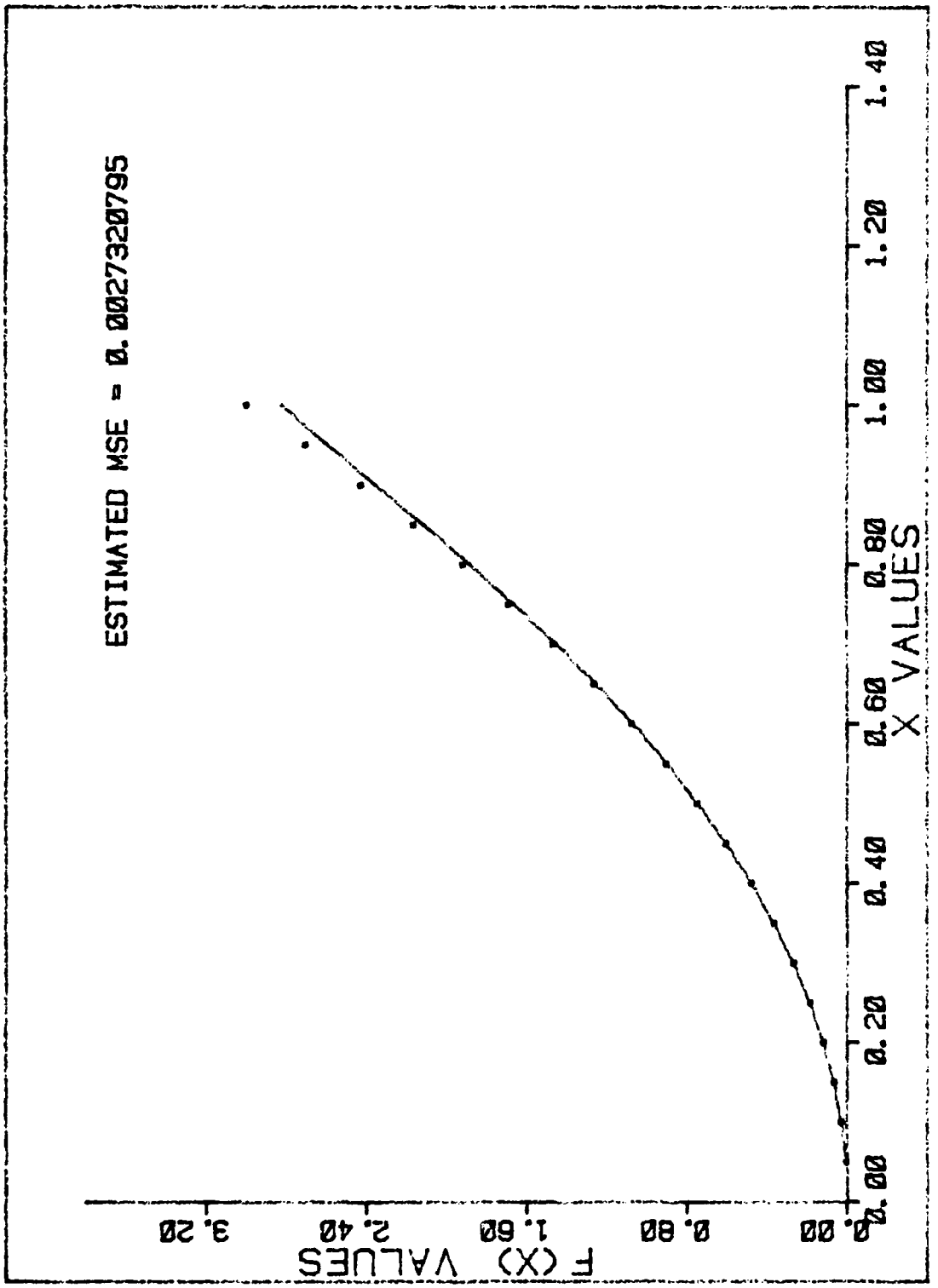

Figure 12A. Power Function (θ=3) n=20

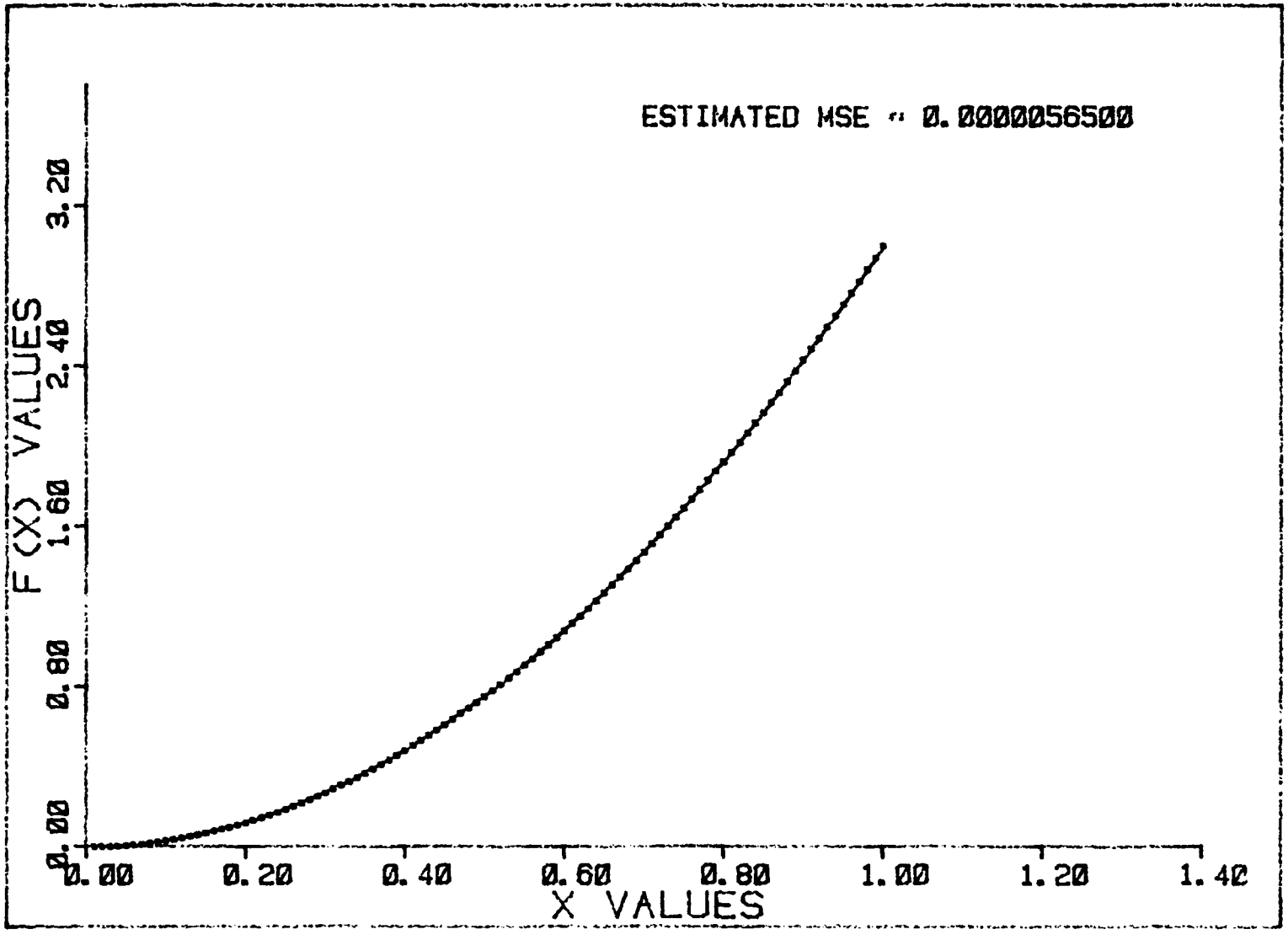

Figure 12B. Power Function (θ=3) n=100

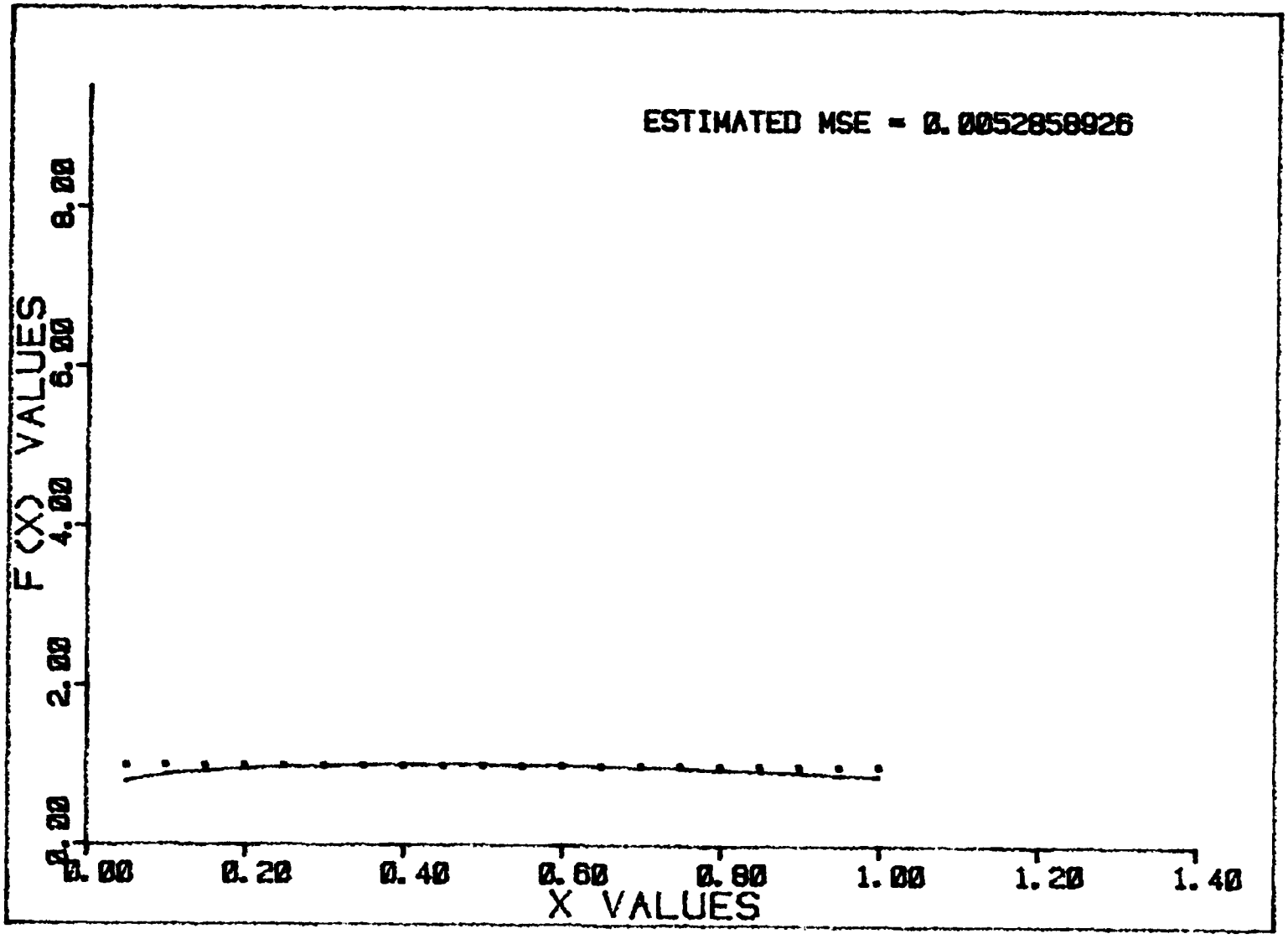

Figure 13A. Uniform n=20

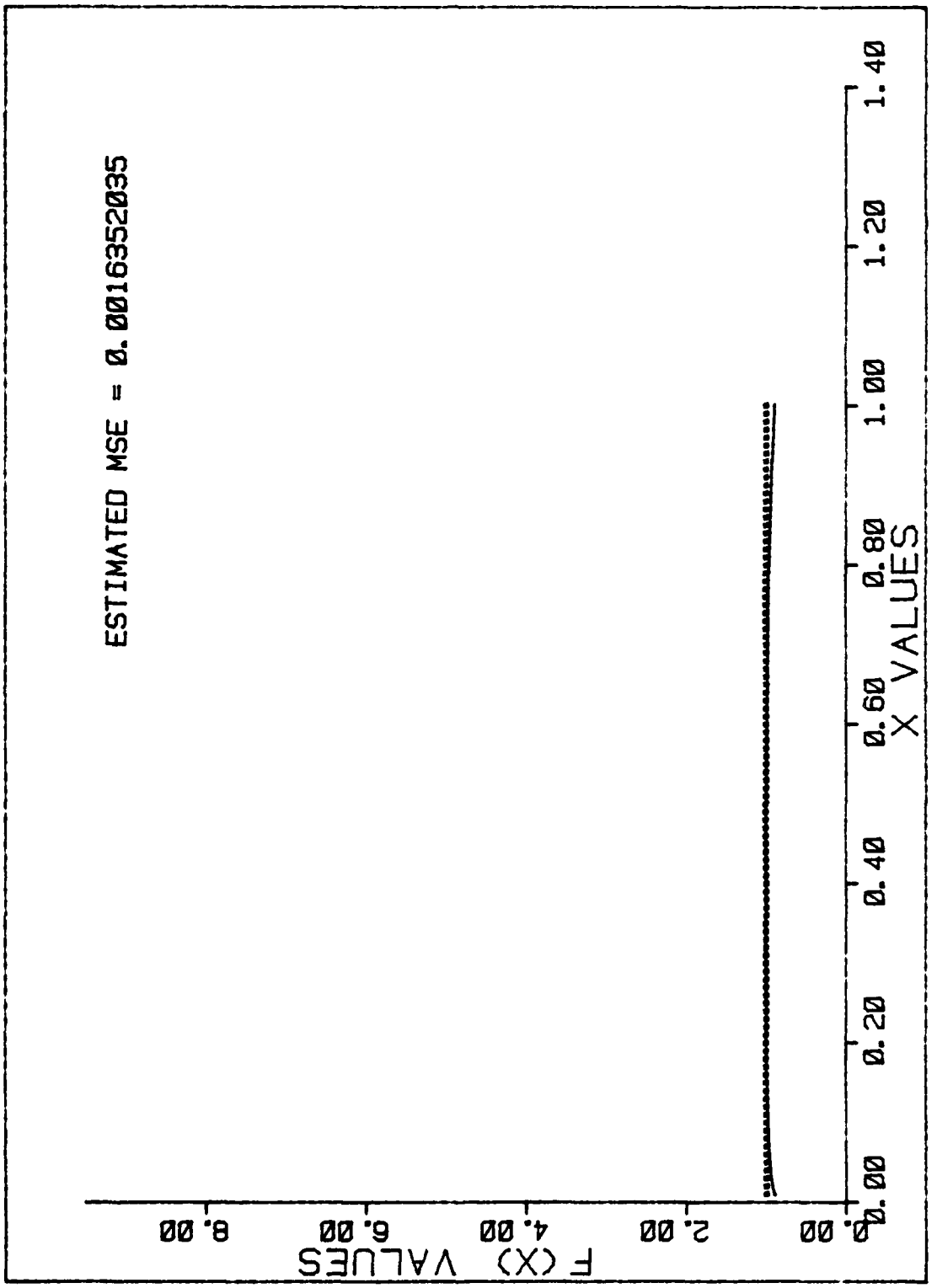

Figure 13B. Uniform n=100

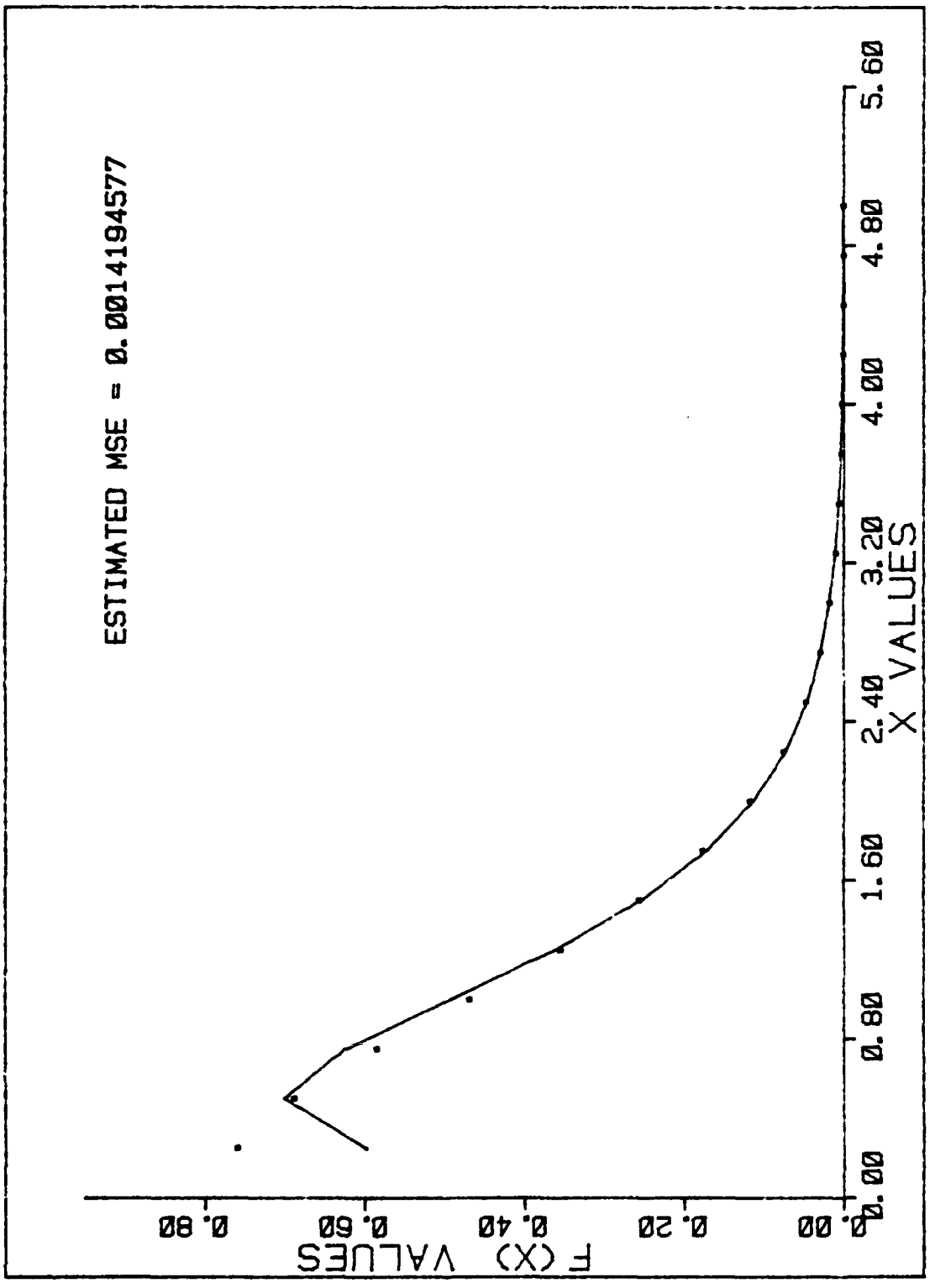

Figure 14A. Half-Student (θ=16) n=20

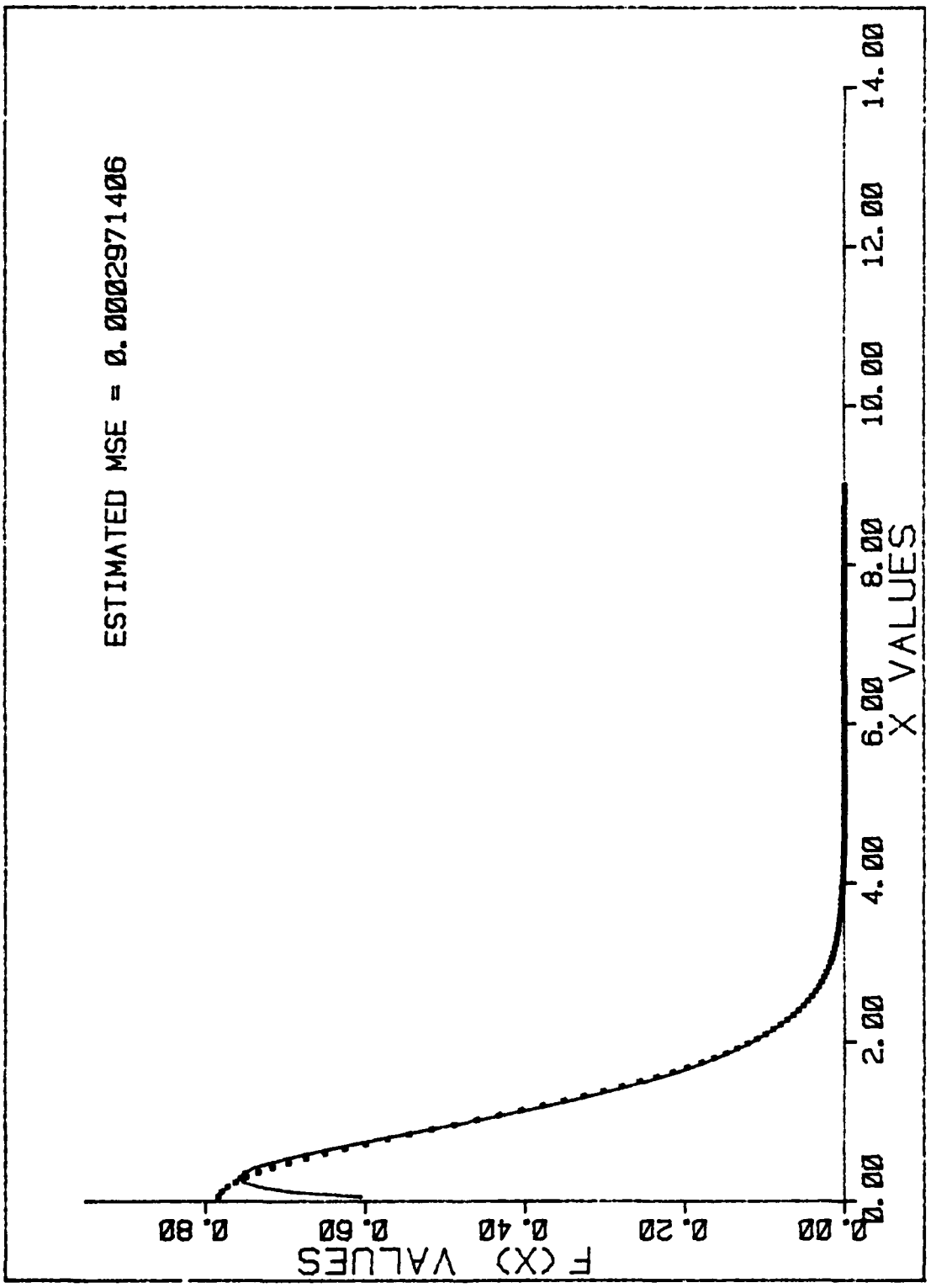

Figure 14B. Half-Student (θ=16) n=150

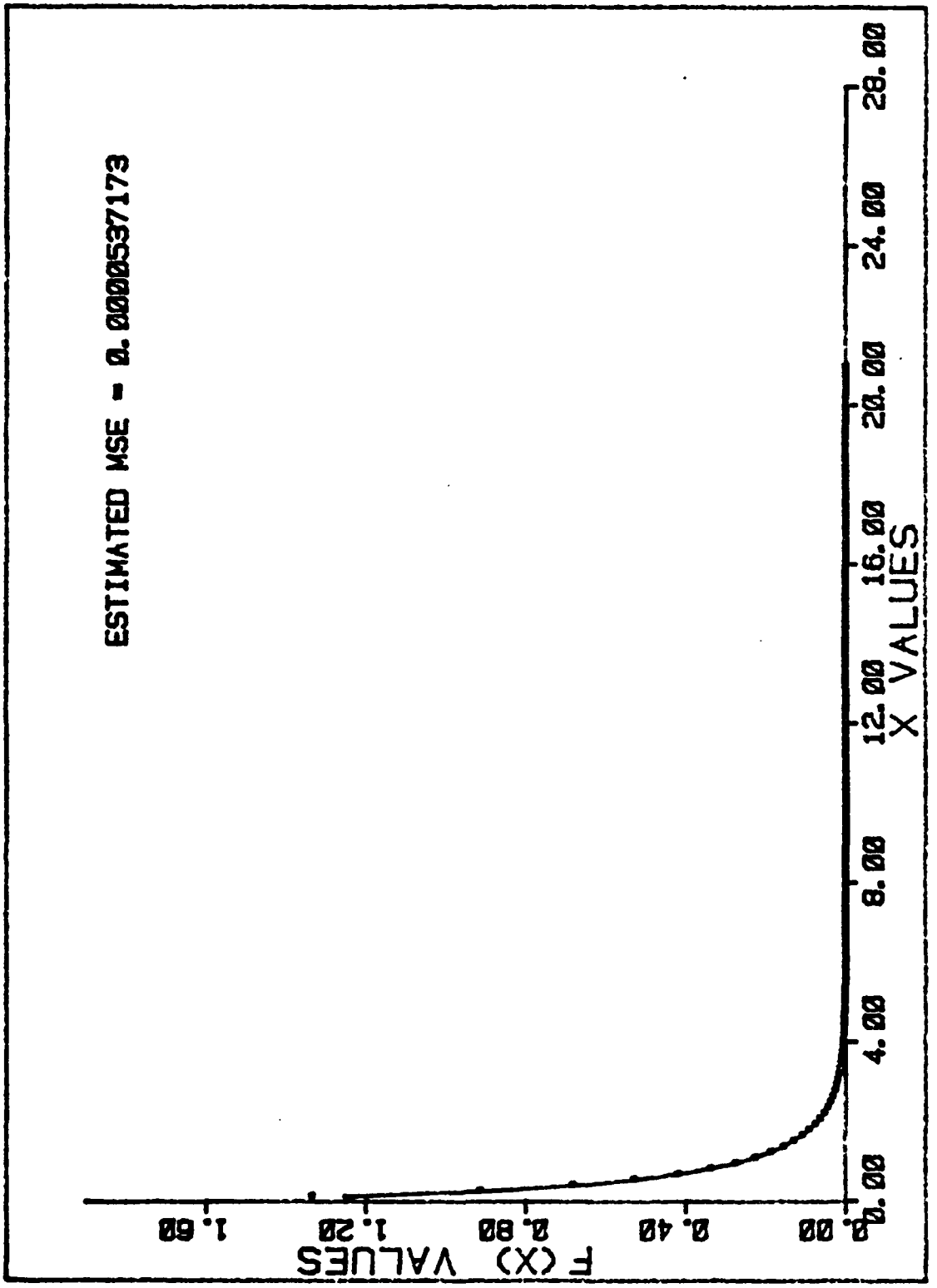

Figure 15. Bessel (θ=1, Ø=1) n=150

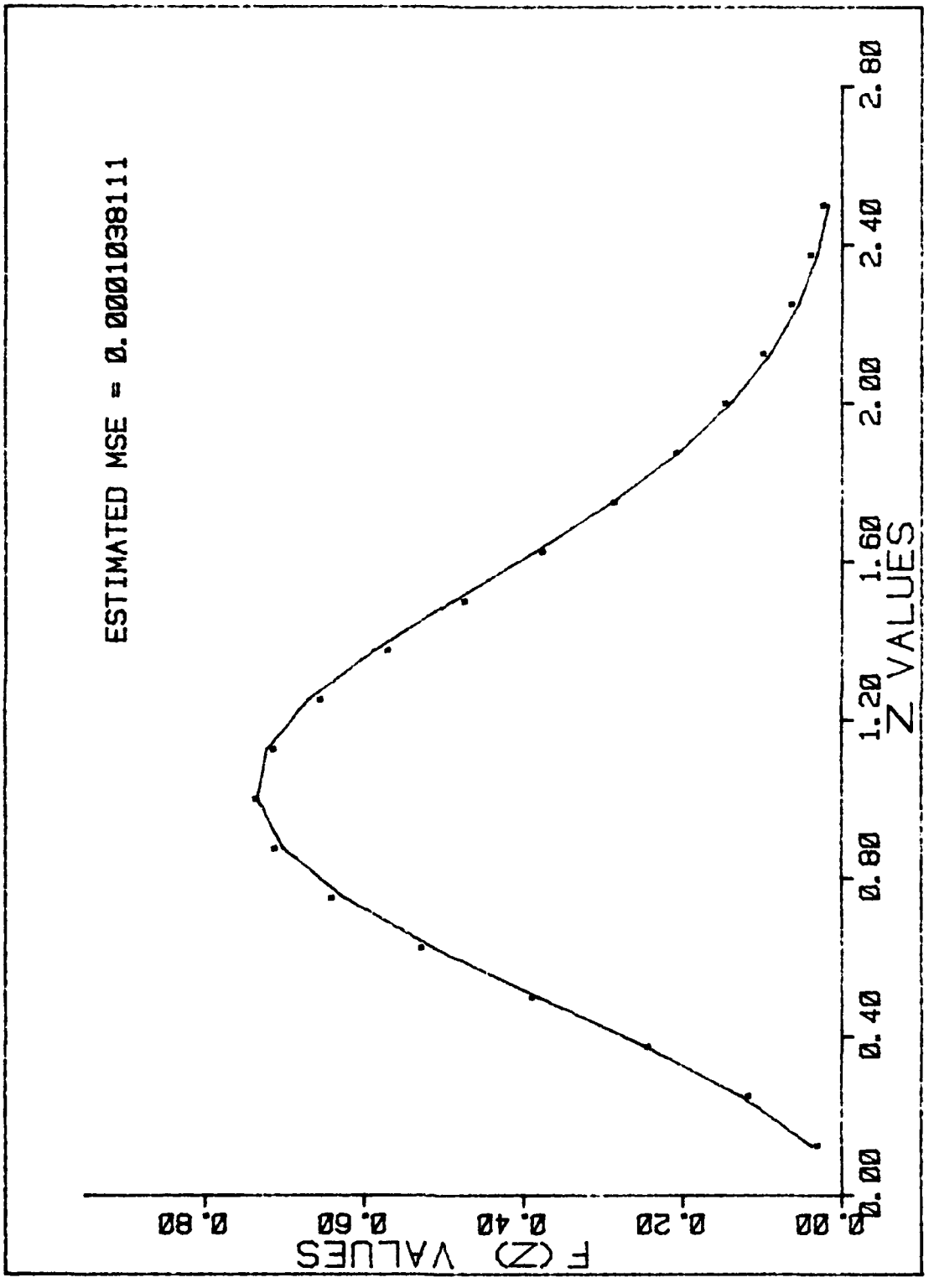

Figure 16A. $2z^2e^{-z^2}$ n=20

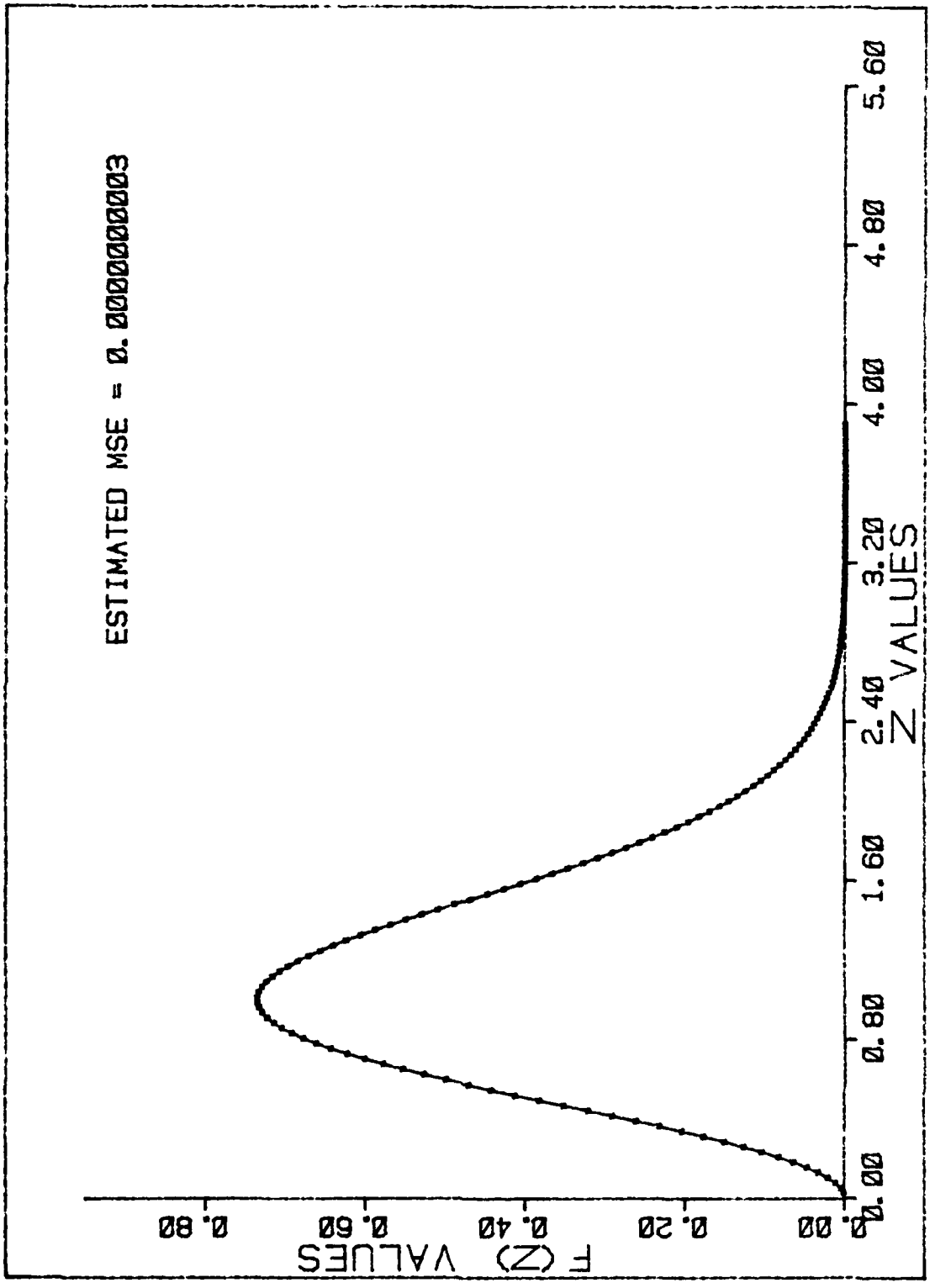

Figure 16B. $2z^2e^{-z^2}$ n=150

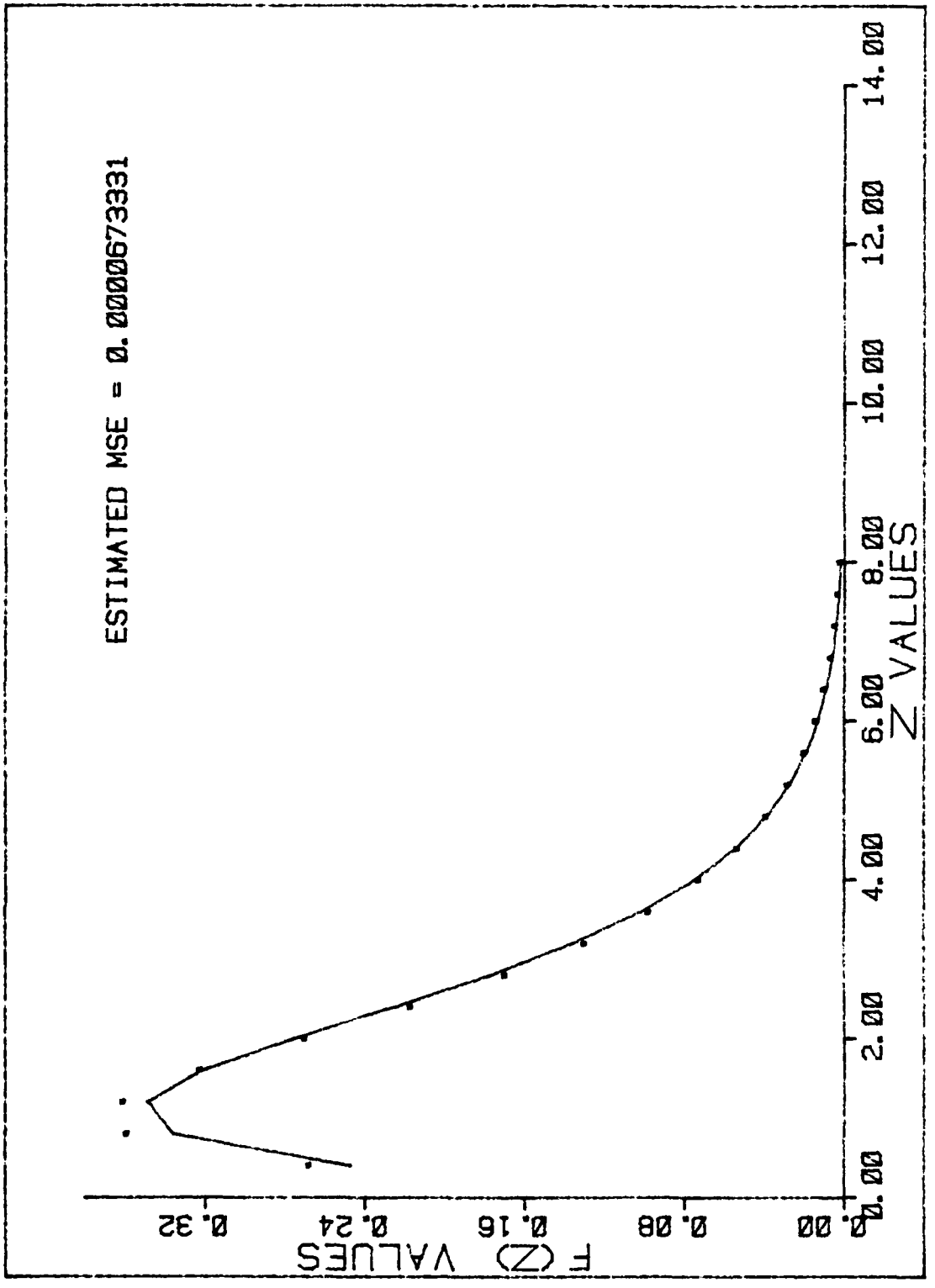

Figure 17A. ze^{-z} n=20

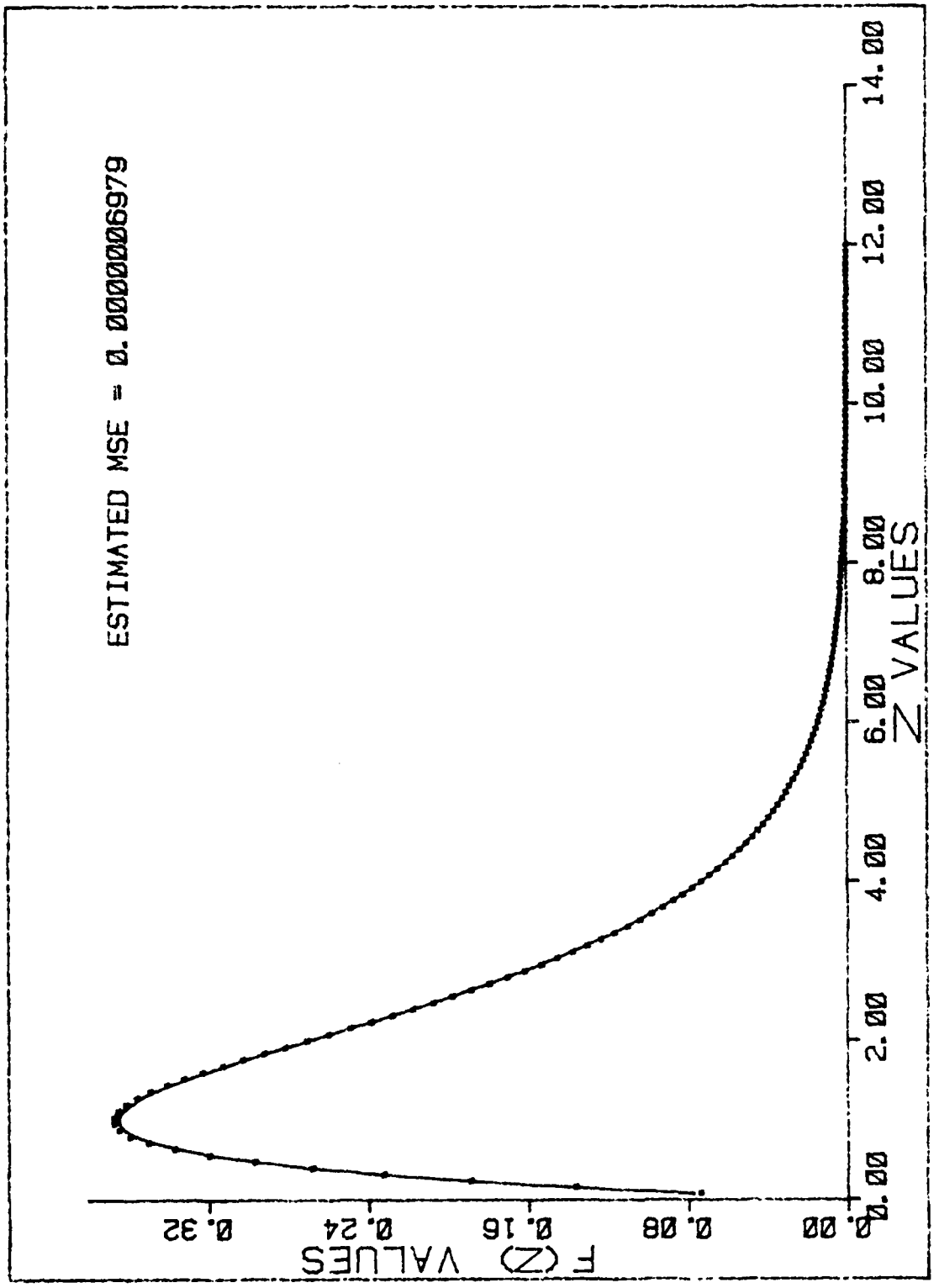

Figure 17B. ze^{-z} n=150

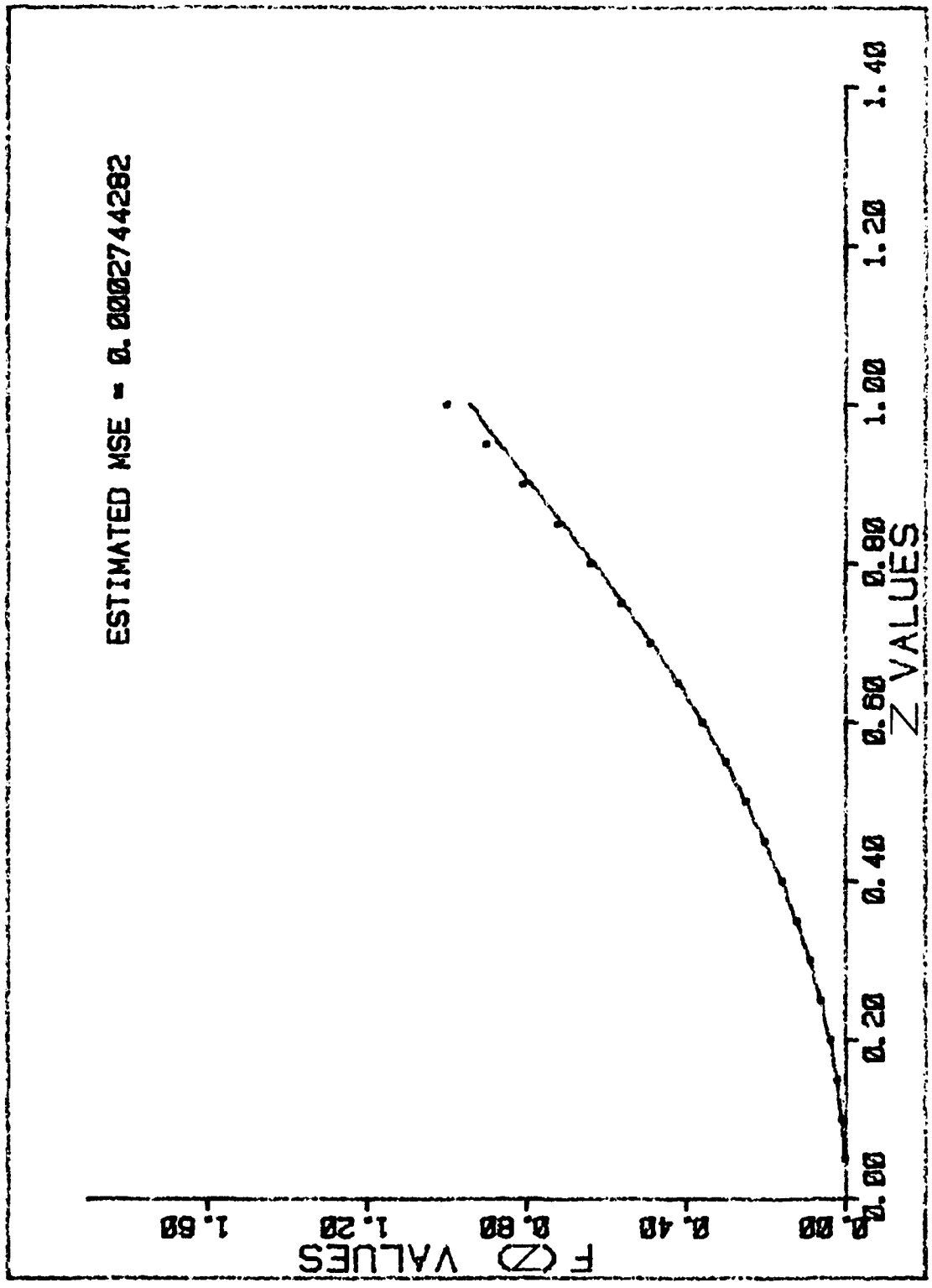

Figure 18A. z^2 n=20

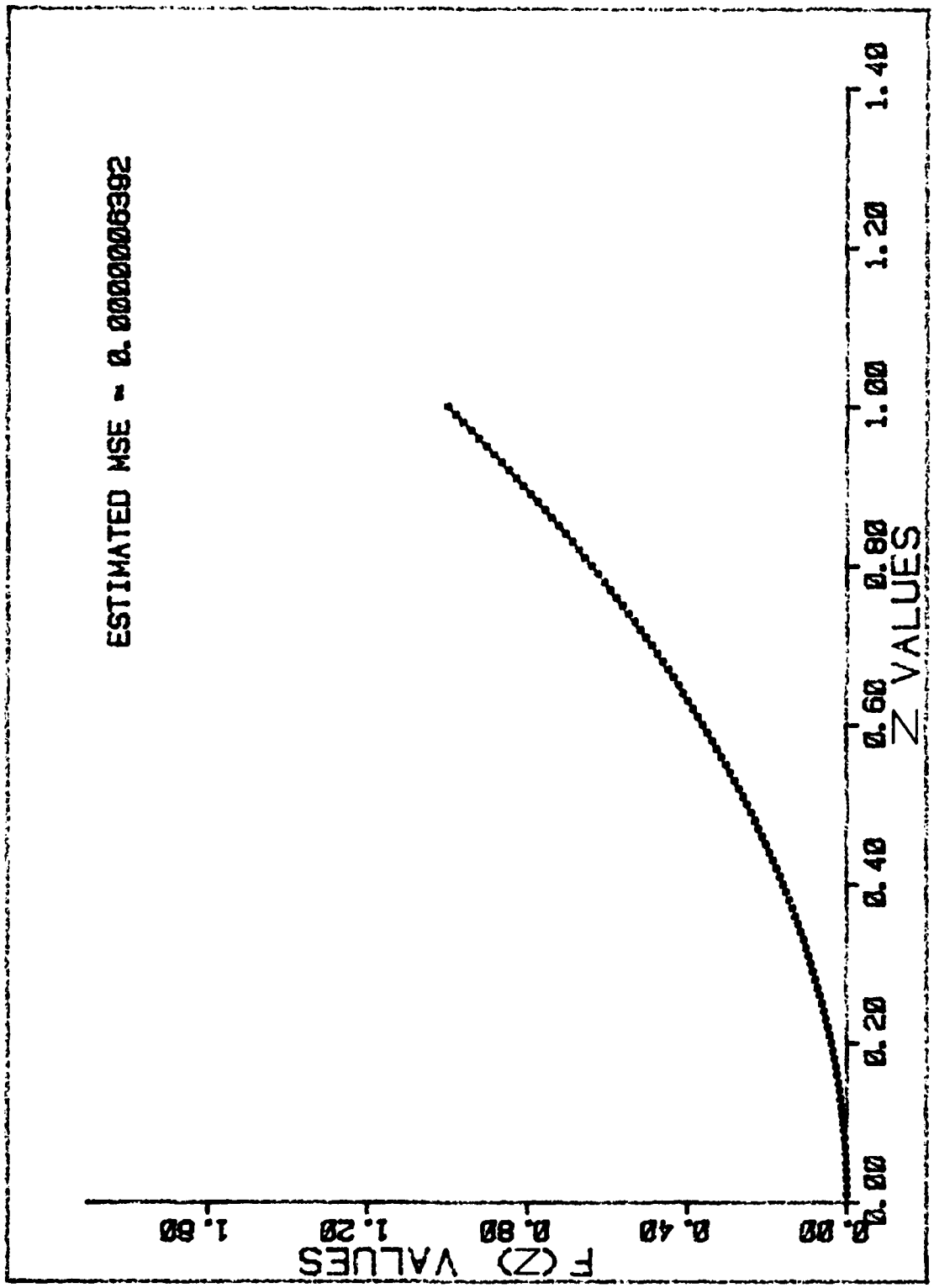

Figure 18B. z^2 n=100

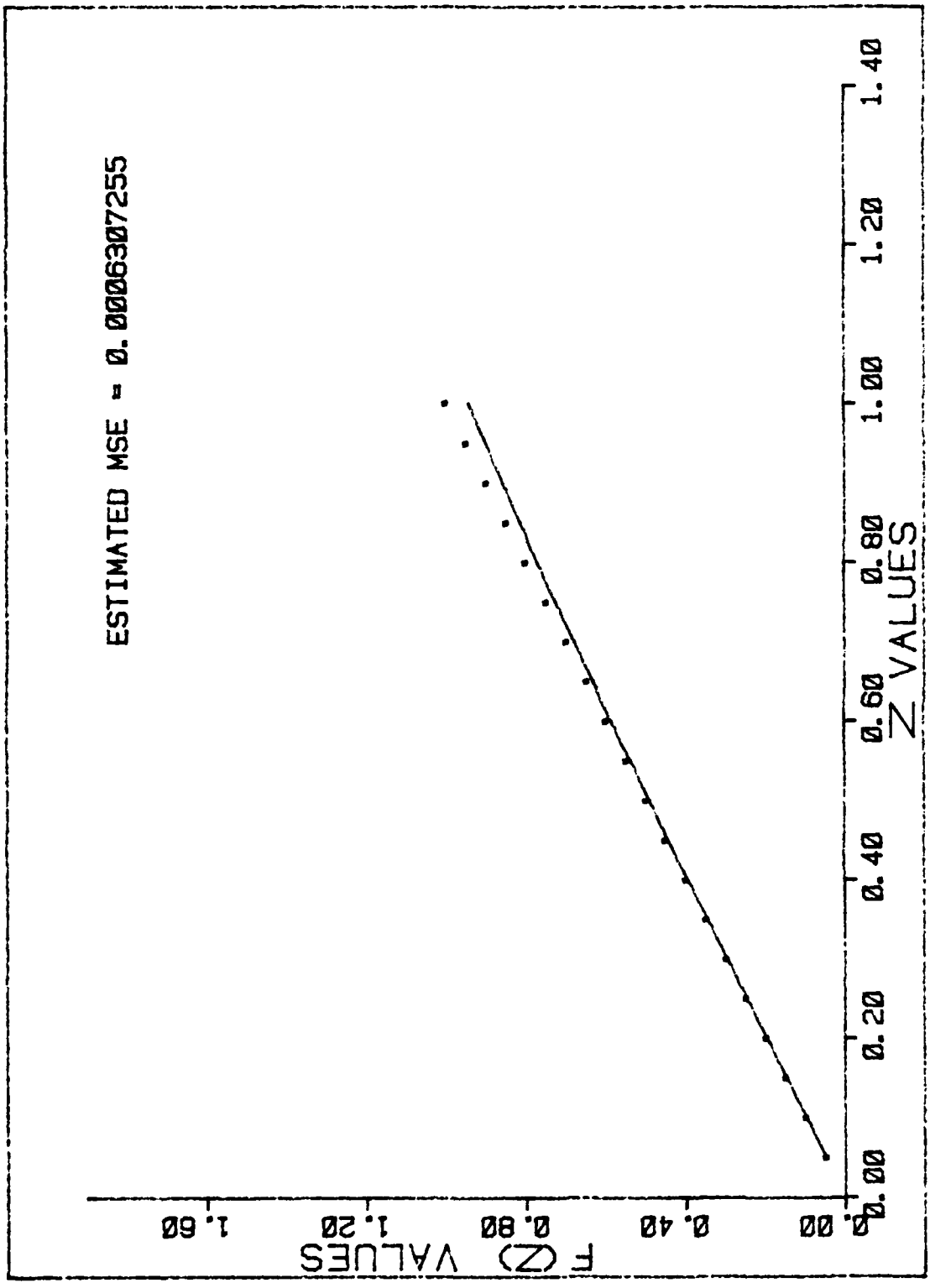

Figure 19A. z n=20

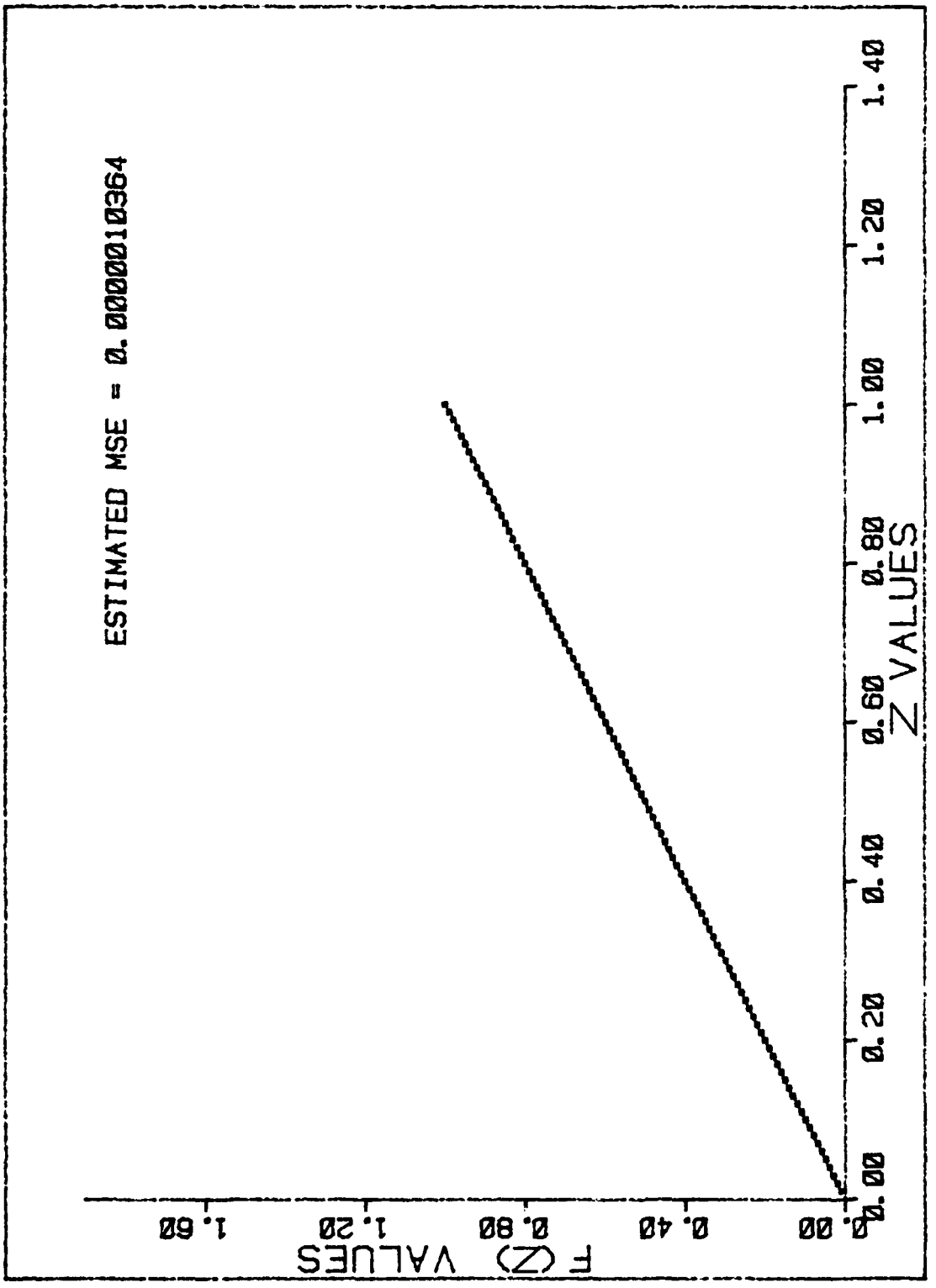

Figure 19B. z n=100

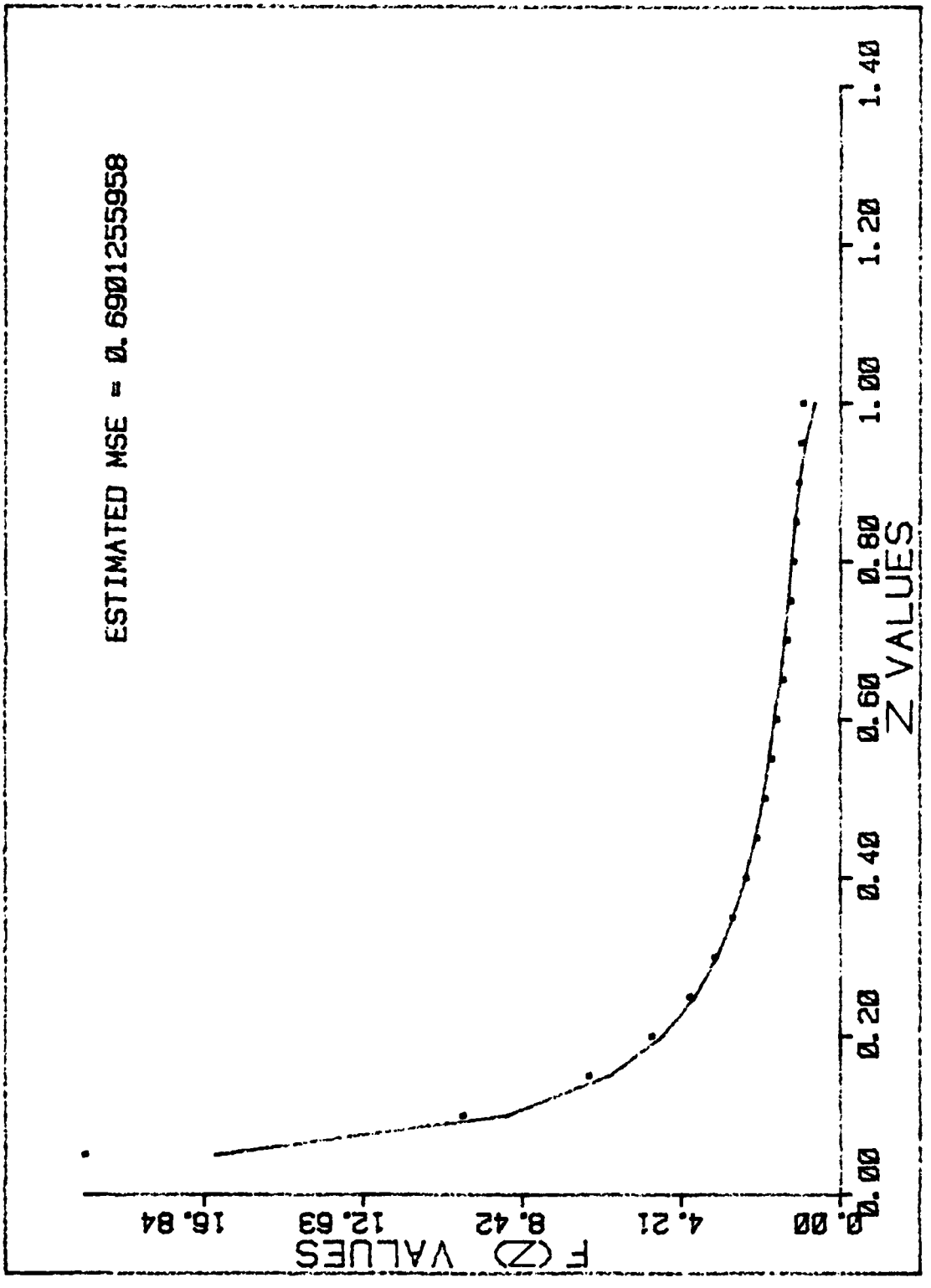

Figure 20A. $\frac{1}{z}$ n=20

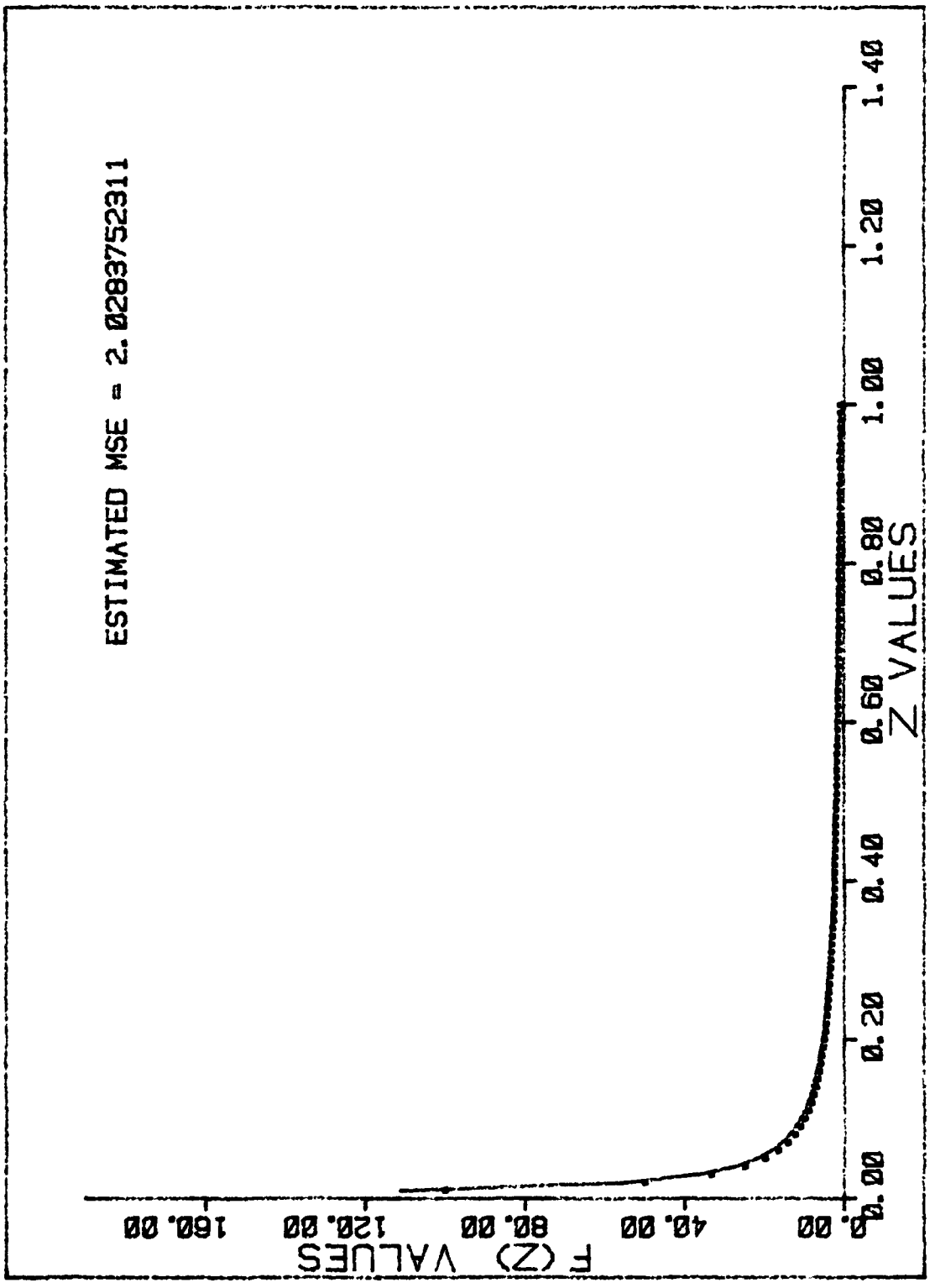

Figure 20B. $\frac{1}{z}$ n=100

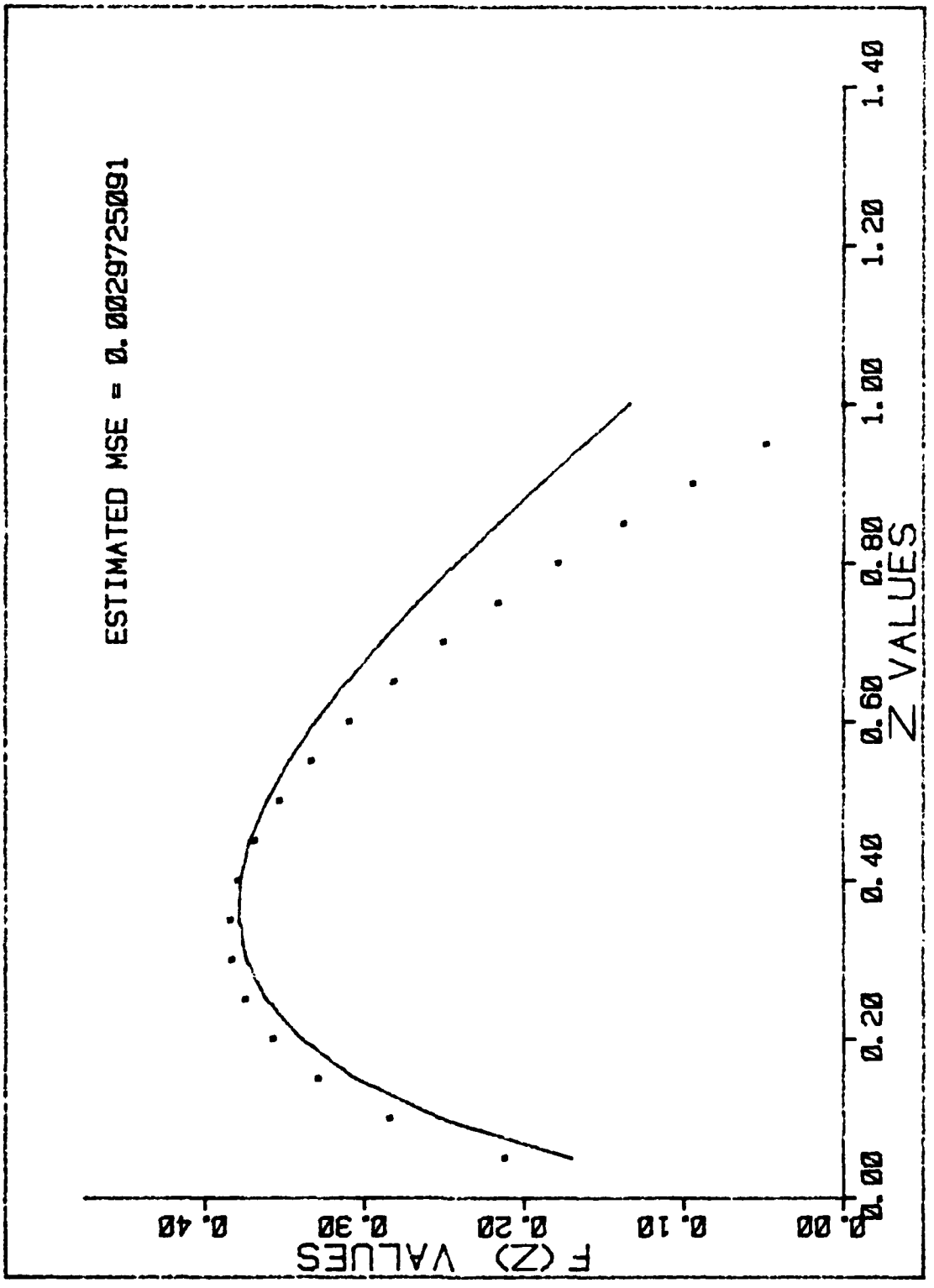

Figure 21A. $\sqrt{z}(1-z)$ n=20

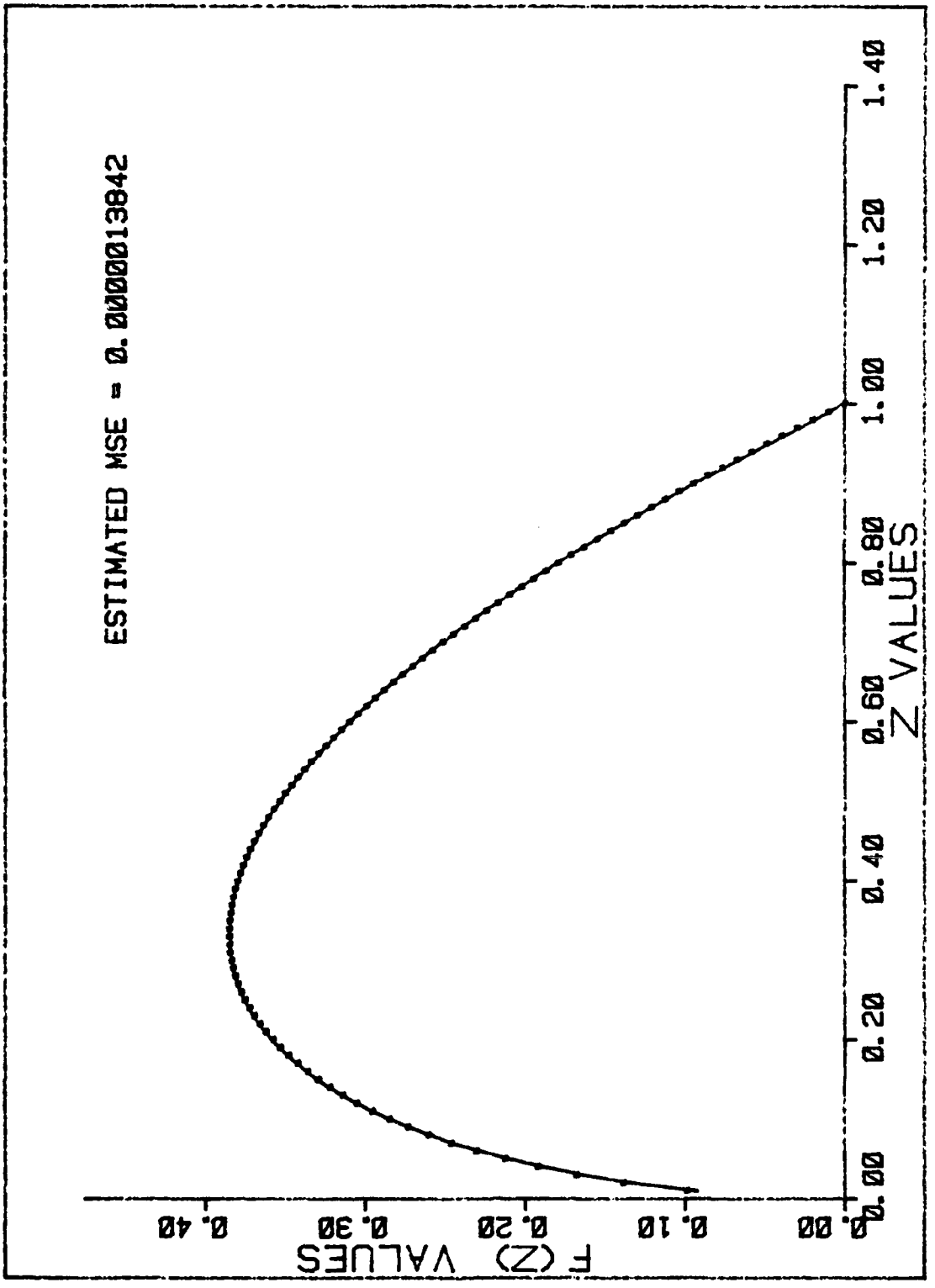

Figure 21B. $\sqrt{z}(1-z)$ n=100

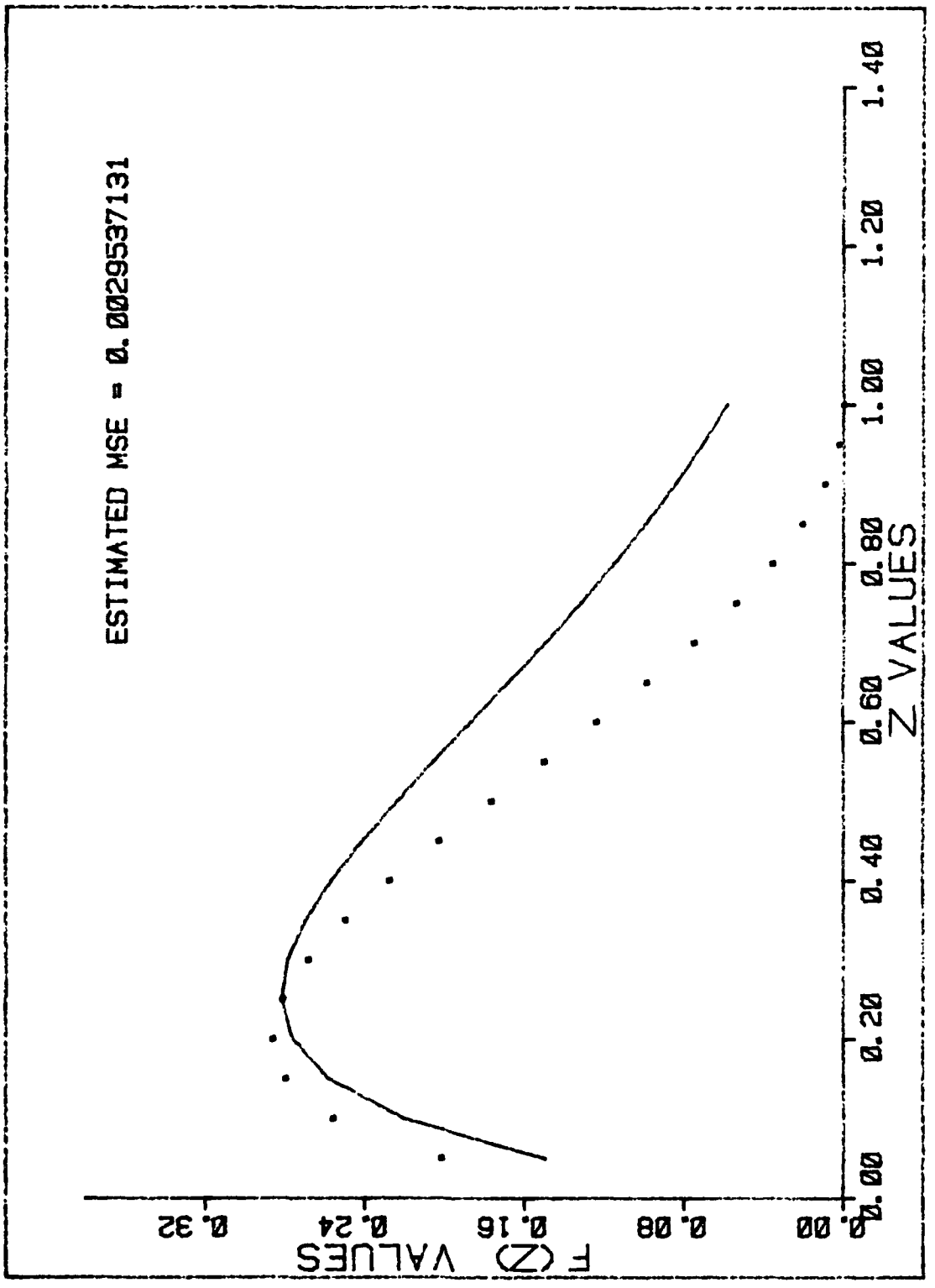

Figure 22A. $\sqrt{z}(1-z)^2$ n=20

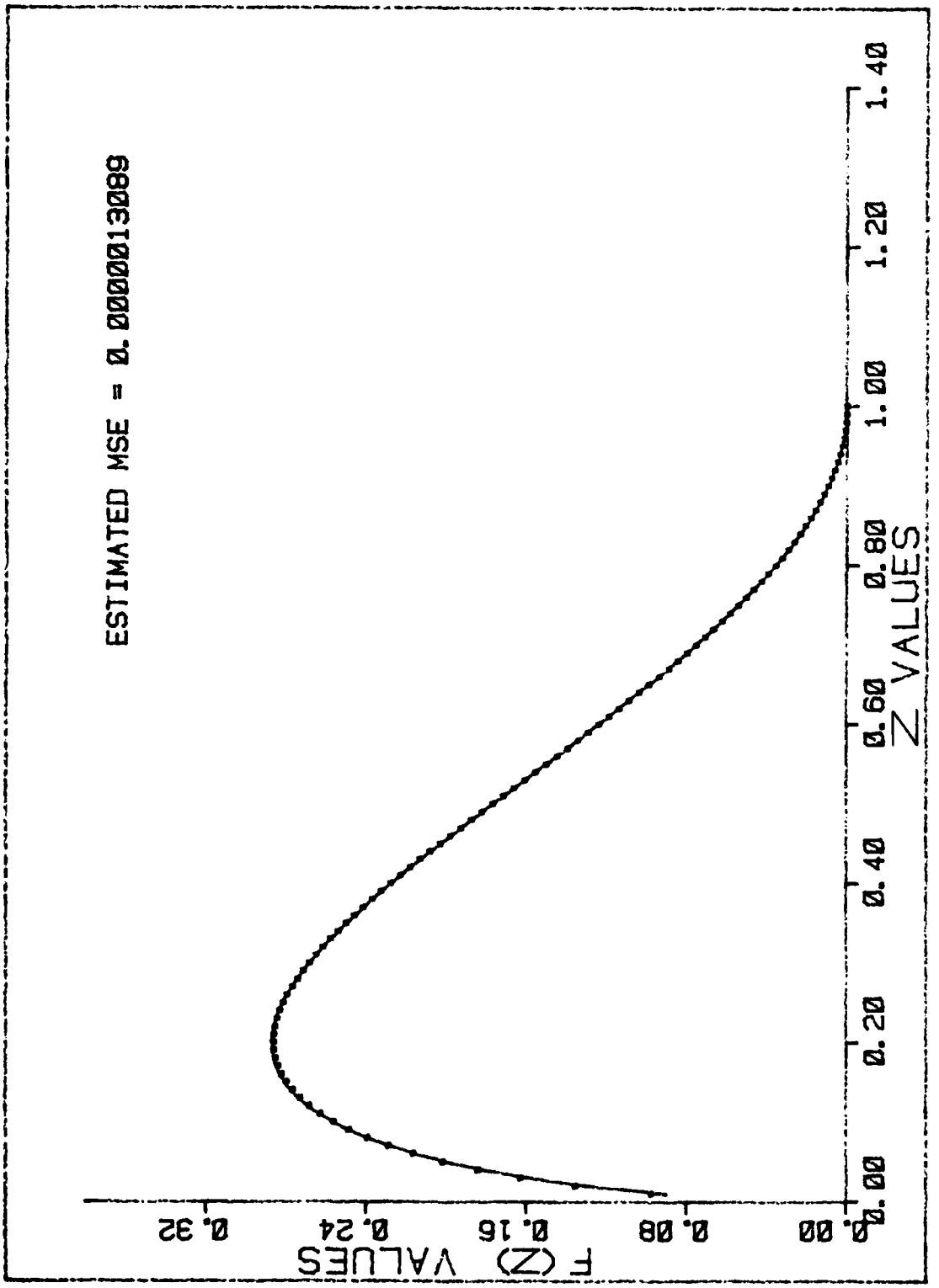

Figure 22B. $\sqrt{z}(1-z)^2$ n=100

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Vita

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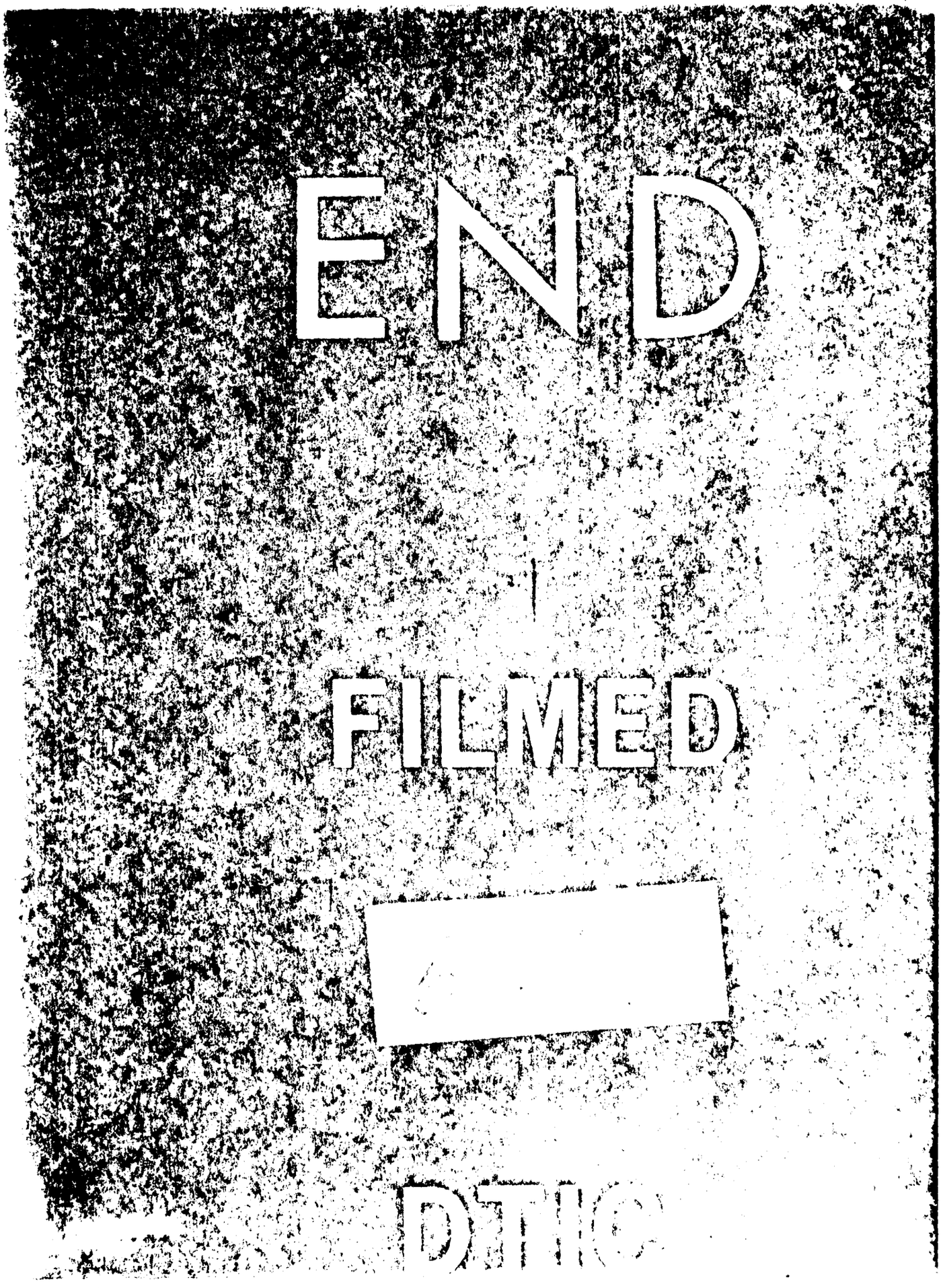
The H-function is the most general special function, encompassing as specific cases many mathematical functions and nearly every continuous statistical distribution defined over positive x. A general procedure is developed to estimate the parameters of the H-function which gives the best fit to a set of data. The technique creates a system of nonlinear equations from the method of moments and uses Powell's quasi-Newton hybrid algorithm to solve the equations. A computer program, which can accept both raw data or previously calculated moments, implements the general process. Several new theoretical results are also presented.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